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THE COMPLETENESS OF THE ORTHOGONAL SYSTEM OF THE HOUGH FUNCTIONS

BY

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Translation of

Die Vollständigkeit des Orthogonalsystems der Hough-Funktionen

Von Peter Holl

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Translator's Preface

Dr. Holl's paper on the completeness of the orthogonal system of the Hough functions was published in the Nachrichten der Akademie der Wissenschaften in Göttingen, II. Mathematisch-Physikalische Klasse, for the year 1970 (no. 7), pp. 159-168. The increasing importance of the Hough functions in theoretical meteorology and oceanography has led to numerous requests for an English translation which is given in the following pages.

Dr. Holl has kindly read and, where necessary, corrected my translation. Thanks are due to Dr. Holl and Prof. H. H. Voigt, the present President of the Academy of the Sciences in Göttingen, for permitting the publication of this translation. The assistance of Prof. M. Siebert, Director of the Institute of Geophysics at the University of Göttingen, and of Prof. T. Vonder Haar, Head of the Department of Atmospheric Science of Colorado State University, in its publication is gratefully acknowledged.

Bernhard Haurwitz

March, 1979

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SUMMARY

It is shown that the differential equation of Hough functions has simple real eigenvalues with the only limit-point at infinity for any given combination of parameters. The eigenfunctions form a complete orthogonal system in the Hilbert space of real-valued square-summable functions $L_2(-1, +1)$. In certain cases, physically corresponding to oscillations with periods less than one-half sidereal day, the eigenvalues are non-negative.

INTRODUCTION AND FORMULATION OF THE PROBLEM

The solutions of Laplace's tidal equation for the rotating spherical earth (Hough functions) have for a long time played only a secondary role in the theoretical investigations of the tides. This was, on the one hand, due to the difficulties in computing the functions numerically with sufficient accuracy; on the other hand, the erroneous opinion existed that the general behavior of the solutions could be deduced from the known properties of Sturm-Liouville systems. An additional reason was the limited applicability of the Hough functions in the computation of oceanic tides. However, during the last twenty years, a large field of applications developed for the Hough functions in connection with atmospheric tides. This subject became even more important when the importance of tidal phenomena in the high atmosphere was recognized.

Hough (1) was the first to succeed in integrating Laplace's tidal equation by series of spherical surface harmonics, after other methods had been found to be either unsatisfactory or unsuccessful. Therefore, Siebert (2,2a) called these functions, which thus had become amenable to numerical calculations, Hough functions, a name which is today generally used. They have been recently studied theoretically and also computed numerically extensively by Flattery (3) and Longuet-Higgins(4).

The Hough functions satisfy the following differential equation:

$$\frac{d}{d\mu}\left(\frac{1-\mu^2}{f^2-\mu^2}\,\frac{d\theta}{d\mu}\right)-\frac{1}{f^2-\mu^2}\left(\frac{l^2}{1-\mu^2}+\frac{l}{f}\,\frac{f^2+\mu^2}{f^2-\mu^2}\right)\theta+\beta\theta=0, \tag{1}$$
$$-1\leqq\mu\leqq+1,\quad 0<|f|<\infty,\quad l=0,1,2,\ldots$$

The detailed derivation of this equation and its connection with the theory of atmospheric tides is discussed by Kertz (5) and Siebert (6). The parameter ℓ determines the zonal dependence of the complete solution of the tidal equations on the sphere. The quantity f is essentially the frequency of the oscillations. If ℓ and f are given, the numbers β are eigenvalues to be determined, if one requires that the solutions θ of equation (1) in the closed interval $I = [-1, +1]$ are finite. The eigenfunctions θ form an orthogonal system for fixed values of ℓ and f, as for instance shown by Flattery (3).

The question which remained unanswered in the previous investigations is whether the system of Hough functions is also complete. In fact, Lindzen (7) is of the opinion that this is not the case. The familiar theorems of the theory of self-adjoint eigenvalue problems cannot be applied to equation (1) because the coefficients of the differential equation have singularities at the limit points of the interval I and possibly also in its interior.

It is possible to prove the completeness of the system of Hough functions in the following manner. The eigenvalue problem is changed, by means of the formalism of the Green's function, into an integral equation; and it is shown that the integral operator which appears is linear, self-adjoint, and completely continuous in the Hilbert space of functions which are square-integrable over the interval I. We consider the following parameter combinations:

① $|f| > 1,\ l \neq 0$; ② $|f| < 1,\ l \neq 0$; ③ $|f| = 1,\ l \neq 0$;
④ $|f| \neq 1,\ l = 0$; ⑤ $|f| = 1,\ l = 0$.

CASE 1

Since $|f| > 1$, no singularities appear in the interior of I. We consider the singular behavior of the solutions at the points $\mu = \pm 1$. We have here, according to the familiar theorems of the theory of differential equations, regular singular points. The exponent of the singularities at $\mu = \pm 1$ is $\alpha = \pm \frac{\ell}{2}$. The solutions can be expanded into the following series in the vicinity of $\mu = +1$ and $\mu = -1$:

(2a)
$$\begin{cases} W_1^{(1)} = (\mu - 1)^{l/2} \sum_{K=0}^{\infty} a_K (\mu - 1)^K \\ W_2^{(1)} = A \cdot W_1^{(1)} \log(\mu - 1) + (\mu - 1)^{-l/2} \sum_{K=0}^{\infty} b_K (\mu - 1)^K \end{cases}$$

and:

(2b)
$$\begin{cases} W_1^{(-1)} = (\mu + 1)^{l/2} \sum_{K=0}^{\infty} \bar{a}_K (\mu + 1)^K \\ W_2^{(-1)} = A \cdot W_1^{(-1)} \log(\mu + 1) + (\mu + 1)^{-l/2} \sum_{K=0}^{\infty} \bar{b}_K (\mu + 1)^K. \end{cases}$$

These functions are regular in the interior of I. Their behavior at the boundary points $\mu = \pm 1$ is as follows: $W_1^{(1)}$ is proportional to $(\mu-1)^{\ell/2}$ at $\mu = +1$, thus also regular. For the point $\mu = -1$, two possibilities exist: (a) $W_1^{(1)}$ is regular at $\mu = -1$. This will be denoted as an exceptional case and considered separately. (b) $W_1^{(1)}$ is irregular at $\mu = -1$. Then it can be written as a linear combination of $W_1^{(-1)}$ and $W_2^{(-1)}$. At $\mu = +1$ the sum $\sum_{K=0}^{\infty} a_K(\mu-1)^K$ behaves as $(\mu+1)^{-\ell/2}$. Thus we have:

$$W_1^{(1)} = \left(\frac{\mu-1}{\mu+1}\right)^{l/2} r_1(\mu), \tag{3a}$$

where r_1 is continuous in I. $W_1^{(-1)}$ is regular at $\mu = -1$. If this were also true at $\mu = +1$, $W_1^{(-1)}$ would be proportional to $W_1^{(1)}$; but this is impossible since $W_1^{(1)}$ has a singularity at $\mu = +1$. Thus at $\mu = +1$ the sum $\sum_{K=0}^{\infty} \bar{a}_K(\mu+1)^K$ behaves as $(\mu-1)^{-\ell/2}$. It follows that

$$W_1^{(-1)} = \left(\frac{\mu+1}{\mu-1}\right)^{l/2} r_2(\mu), \tag{3b}$$

where r_2 is also continuous in I.

By means of $W_1^{(1)}$ and $W_1^{(-1)}$ the Green's function G of the eigenvalue problems can now be constructed in the usual way. G is symmetric in the arguments since the problem is self-adjoint:

$$G(\mu, \hat{\mu}) = \begin{cases} \left(\frac{(\mu-1)(\hat{\mu}+1)}{(\mu+1)(\hat{\mu}-1)}\right)^{l/2} r_1(\mu)\, r_2(\hat{\mu}), & \mu \geqq \hat{\mu} \\ \left(\frac{(\mu+1)(\hat{\mu}-1)}{(\mu-1)(\hat{\mu}+1)}\right)^{l/2} r_2(\mu)\, r_1(\hat{\mu}), & \mu \leqq \hat{\mu} \end{cases} \tag{4}$$

With the aid of G the eigenvalue problem may be formulated as an integral equation:

$$\beta \cdot \int_{-1}^{+1} G(\mu, \hat{\mu}) \cdot \theta(\hat{\mu})\, d\hat{\mu} = \theta(\mu) = \beta \mathfrak{K}(\theta). \tag{5}$$

Within the meaning of functional analysis, we consider (5) as operator equation in the Hilbert space of functions which are square-integrable over I. It requires no explanation that the integral operator $\mathfrak{K}$ is linear and self-adjoint. We shall show that $\mathfrak{K}$ is completely continuous. To prove this we have to show {cf. Schmeidler (8)}, that the integral

$$\|\mathfrak{K}^2\| = \int_{-1}^{+1}\int_{-1}^{+1} |G(\mu,\hat{\mu})|^2 \, d\mu \, d\hat{\mu} \tag{6}$$

does exist. With equation (4) the norm of the operator $\mathfrak{K}$ in equation (5) becomes:

$$\|\mathfrak{K}^2\| = \int_{-1}^{+1} d\hat{\mu}\left[r_1^2(\hat{\mu})\cdot\left(\frac{\hat{\mu}-1}{\hat{\mu}+1}\right)^l \cdot \int_{-1}^{\hat{\mu}} r_2^2(\mu)\cdot\left(\frac{\mu+1}{\mu-1}\right)^l d\mu \right.$$
$$\left. + r_2^2(\hat{\mu})\left(\frac{\hat{\mu}+1}{\hat{\mu}-1}\right)^l \int_{\hat{\mu}}^{+1} r_1^2(\mu)\left(\frac{\mu-1}{\mu+1}\right)^l d\mu\right].$$

The mean value theorem of integral calculus is applicable to the interior integrals. We obtain:

$$\|\mathfrak{K}^2\| = \frac{1}{l+1}\int_{-1}^{+1} d\hat{\mu}\left[r_1^2(\hat{\mu})\cdot r_2^2(\mu_0(\mu))\left(\frac{(\hat{\mu}+1)(\hat{\mu}-1)}{\mu_0(\hat{\mu})-1}\right)^l \right.$$
$$\left. + r_2^2(\hat{\mu})\cdot r_1^2(\mu_1(\hat{\mu}))\left(\frac{(\hat{\mu}-1)(\hat{\mu}+1)}{\mu_1(\hat{\mu})+1}\right)^l\right], \tag{7}$$
$$-1 \leqq \mu_0 \leqq \hat{\mu}, \quad \hat{\mu} \leqq \mu_1 \leqq +1.$$

The integrand is continuous in I. Consequently, $\|\mathfrak{K}^2\|$ does exist. Now the following theorem holds {cf. Schmeidler (8)}:

> Every completely continuous self-adjoint linear operator not identically zero possesses a finite or infinite number of real eigenvalues β_n with a finite multiplicity. The β_n have no finite point of accumulation. The associated eigenvectors θ_n form an orthogonal system. This orthogonal system can be extended to a complete orthogonal system in Hilbert space by the addition of the null solutions.

As we saw, in the case considered here the Green's function does exist. Therefore the set of zero solutions is empty.

All the eigenvalues are simple: otherwise there would be two linearly independent solutions, which would be regular in the interval I. But this would be a contradiction to the fact that there is always a solution with a singularity at $\mu = +1$ or $\mu = -1$.

We consider now the exceptional case (a): i.e., the possibility that $W_1^{(1)}$ is regular at $\mu = -1$. Then $W_2^{(1)}$ has to be irregular at $\mu = -1$ because there must be a solution which is irregular at $\mu = -1$, as shown by equation (2). This implies two things. First, $\beta = 0$ is an eigenvalue. Second, there is no Green's function for the eigenvalue problem. However, according to Courant-Hilbert (9) a Green's function of a more generalized type can be constructed. This is done by determining the Green's function not from the homogeneous differential equation (1), but from the following equation:

$$\frac{d}{d\mu}\left(\frac{1-\mu^2}{f^2-\mu^2}\,\frac{dG^*}{d\mu}\right)-\frac{1}{f^2-\mu^2}\left(\frac{l^2}{1-\mu^2}+\frac{l}{f}\,\frac{f^2+\mu^2}{f^2-\mu^2}\right)G^* = W_1^{(1)}(\mu)\,W_1^{(1)}(\hat{\mu}). \tag{8}$$

G^* is determined uniquely if we postulate that

$$\int_{-1}^{+1} G^*(\mu,\hat{\mu})\,W_1^{(1)}(\mu)\,d\mu = 0 \tag{9}$$

holds together with the usual continuity and discontinuity postulates. Our eigenvalue problem can now be formulated as an integral equation which has a form analogous to equation (5).

Since the problem (1) is self-adjoint, it can be shown {for instance, Courant-Hilbert (9)}, that G^* is also symmetrical in the arguments.

In our case $W_1^{(1)}$ can be written in the following form:

$$W_1^{(1)} = [(\mu-1)(\mu+1)]^{l/2}\,r_1(\mu), \tag{10}$$

where r_1 is continuous in I. Evidently it is true that:

$$W_2^{(1)} = C_1\,W_1^{(-1)} + C_2\,W_2^{(-1)}.$$

Consequently, $W_2^{(1)}$ permits the following representation:

(11) $$W_2^{(1)} = (\mu - 1)^{-1/2} r_2(\mu) + (\mu + 1)^{-1/2} \tilde{r}_2(\mu),$$

with r_2 and $\tilde{r}_2$ being continuous functions in I. The Wronskian of a fundamental system of solutions of the homogeneous differential equation belonging to equation (8) has the following form because of Abel's identity:

(12) $$W = W_0 \cdot \frac{f^2 - \mu^2}{1 - \mu^2}.$$

The general solution of (8) is obtained by variation of parameters:

(13) $$y = A_1 W_1^{(1)} + A_2 W_2^{(1)} + W_2^{(1)} \cdot \int \frac{W_1^{(1)2}(\mu) \cdot W_1^{(1)}(\hat{\mu})}{W_0} d\mu - W_1^{(1)} \cdot \int \frac{W_1^{(1)}(\mu) \cdot W_1^{(1)}(\hat{\mu}) \cdot W_2^{(1)}(\mu)}{W_0} d\mu.$$

To construct the Green's function G* we choose a solution which is regular at $\mu = -1$:

$$y_- = W_2^{(1)} \cdot \int_{-1}^{\mu} \frac{W_1^{(1)2}(\mu') W_1^{(1)}(\hat{\mu}) d\mu'}{W_0} - W_1^{(1)} \int_{-1}^{\mu} \frac{W_1^{(1)}(\mu') W_1^{(1)}(\hat{\mu}) W_2^{(1)}(\mu')}{W_0} d\mu'.$$

With the aid of the mean value theorem of integral calculus and the equations (10) and (11), we find for y_- the following representation:

(14) $$y_- = \left(\frac{\mu + 1}{\mu - 1}\right)^{1/2} r_-(\mu).$$

As the solution regular at $\mu = +1$ we choose:

$$y_+ = W_1^{(1)} \int_{\mu}^{+1} \frac{W_1^{(1)}(\mu') W_1^{(1)}(\hat{\mu}) W_2^{(1)}(\mu')}{W_0} d\mu' - W_2^{(1)} \cdot \int_{\mu}^{+1} \frac{W_1^{(1)2}(\mu') W_1^{(1)}(\mu')}{W_0} d\mu'.$$

In a similar manner as before, we find

(15) $$y_+ = \left(\frac{\mu - 1}{\mu + 1}\right)^{1/2} r_+(\mu).$$

The functions r_+ and r_- are continuous in I. From (14) and (15) it is apparent that G* has the same form as G in equation (4). The operator which appears in the integral equation is thus completely continuous also in the exceptional case. The theorem quoted above is again applicable. However $\beta = 0$ is an eigenvalue. The appropriate null solution is to be added to the orthogonal system so that it becomes complete. Lindzen (7) did not do that in his arguments presented above and concluded, therefore, that the system is incomplete.

Our method of concluding that the eigenvalues do not degenerate remains valid also in the exceptional case considered presently.

It can be shown {Flattery (3)} that the case $\beta = 0$ occurs only if the following holds:

(16) $$f = \frac{l}{n(n+1)} \qquad n = 1, 2, \ldots .$$

The corresponding null solutions have the form:

(17) $$\theta_0 = \frac{n+1}{n} R^l_n P^l_{n-1}(\mu) + \frac{n}{n+1} R^l_{n+1} P^l_{n+1}(\mu),$$

where $R^\ell_n = \sqrt{\frac{(n+\ell)(n-\ell)}{(2n+1)(2n-1)}}$ and P^ℓ_n are the associated Legendre functions.

To conclude the Case 1 it will be shown that the eigenvalue problem is positive semi-definite. We multiply (1) by θ and integrate from -1 to +1:

(18) $$\frac{1-\mu^2}{f^2-\mu^2}\theta'\theta\Big|_{-1}^{+1} - \int_{-1}^{+1}\left[\frac{1-\mu^2}{f^2-\mu^2}\theta'^2 + \frac{1}{f^2-\mu^2}\left(\frac{l^2}{1-\mu^2} + \frac{l}{f}\,\frac{f^2+\mu^2}{f^2-\mu^2}\right)\theta^2\right]d\mu = -\beta\int_{-1}^{+1}\theta^2\, d\mu .$$

The existence of the integrals follows from (3) and (4). The first term in (18) disappears, also if $|f| > 1$ and $\ell = 0$, as can be seen from the arguments in Case 4. If $f > +1$, then the integrals in (8), having always real values, are not negative. Consequently, $\beta \geqq 0$. This holds also for f <-1 and ℓ = 0. If $\ell \geqq 2$, we can show that

$$K = \int_{-1}^{+1} \frac{\theta^2}{f^2 - \mu^2} \left(\frac{l^2}{1-\mu^2} - \frac{l}{|f|} \frac{f^2+\mu^2}{f^2-\mu^2} \right) d\mu < 0$$

and therefore $\beta > 0$. To see this, consider that according to (3) and (4) the Hough functions for $\ell \neq 0$ may be written:

$$\theta = r(\mu)(1-\mu^2)^{l/2},$$

where r is regular in I. With the aid of the mean value theorem of integral calculus:

$$K = 2\frac{r^2(\mu_0)}{f^2-\mu_0^2}(1-\mu_0^2)^{l-1} \int_{-1}^{+1} \left(l^2 - \frac{l(1-\mu^2)}{|f|} - \frac{2l}{|f|}\mu^2 \frac{1-\mu^2}{f^2-\mu^2} \right) d\mu,$$

$$-1 \leqq \mu_0 \leqq -1.$$

Therefore:

$$K \geqq 2\frac{r^2(\mu_0)}{f^2-\mu_0^2}(1-\mu_0^2)^{l-1} \left(l^2 - \frac{2}{3}\frac{l}{|f|} - \frac{2l}{3|f|} \right).$$

If $\ell \geqq 2$ the last term is positive.

We consider now briefly the case $\ell = 1$. In the vicinity of $\mu = -1$, we assume that the Hough function is developed according to powers of $\varepsilon = \mu + 1$. Then:

$$\theta = C \cdot (1-\mu^2)^{1/2}(1 + a_1(\mu+1) + \cdots).$$

Except for quantities which are small compared to ε, equation (1) becomes:

$$-\beta\theta^2 = \frac{C^2}{(f^2-1)^2}\left[4f^2 + \frac{2}{|f|}(f^2+1)\right] \cdot \varepsilon \leqq 0.$$

Therefore, here too $\beta \geqq 0$.

CASE 2

Since now $|f| < 1$ the differential equation (1) has singularities at the points $\mu = \pm f$ in the interior of the interval I. However, all the solutions are regular there {Flattery (3)}. We have thus merely apparent singularities. Consequently, all the arguments remain valid which we have used in Case I concerning the Green's functions.

Thus, here too Hough functions form a complete orthogonal system together with the null solutions. However, the eigenvalues need no longer be positive. The integrands in equation (18) change their sign. As a matter of fact, many authors, especially Dikii (10), Lindzen (11), and Kato (12), have found negative eigenvalues.

CASE 3

Now $|f| = 1$ and $\ell \neq 0$. The differential equation (1) has the form:

$$\theta'' - \frac{1}{(1-\mu^2)^2}(l^2 \pm l(1+\mu^2))\theta + \beta\theta = 0. \tag{19}$$

The values $\mu = \pm 1$ are again regular singular points. If $f = +1$, the exponents of the singularities are $\alpha_1 = \frac{\ell}{2} + 1$ and $\alpha_2 = -\frac{\ell}{2}$; likewise for $f = -1$, $\alpha_1 = \frac{\ell}{2}$ and $\alpha_2 = 1 - \frac{\ell}{2}$. Considerations analogous to those of Case I can be applied to the present situation. The form of the solutions assumed for the equations (2) is valid here, too, since the exponents α_1 and α_2 differ by an integer. Only in the case of $f = -1$, $\ell = 1$ one has to note that $\alpha_1 = \alpha_2 = \frac{1}{2}$, and therefore the singular behavior of the solutions is characterized by a logarithmic term. The conclusion that the Green's functions G and G* are square-integrable still holds. Therefore, the completeness theorem is applicable.

CASE 4

Here $|f| \neq 1$, $\ell = 0$. The differential equation (1) may now be written

$$\left(\frac{1-\mu^2}{f^2-\mu^2}\,\theta'\right)' + \beta\theta = 0. \tag{20}$$

The exponents of the singularities α_1 and α_2 coincide and are equal to zero. θ = const. is always an eigenfunction; that is $\beta = 0$ is always an eigenvalue. We have to construct a Green's function in the more general sense. The fundamental solutions of the homogeneous differential equation have the form

$$(21)\qquad \begin{cases} W_1 = A \\ W_2 = B\left(\mu + \frac{f^2-1}{2}\log\frac{1+\mu}{1-\mu}\right) + C. \end{cases}$$

The functions required for the construction of G* can be written in analogy to equation (13):

$$(22)\qquad y = \bar{A}\,W_1 + \bar{B}\,W_2 + W_2\int \frac{W_1^2(\mu)\,W_1(\hat{\mu})}{W_0}\,d\mu - W_1\int \frac{W_1(\mu)\,W_2(\mu)\,W_1(\hat{\mu})}{W_0}\,d\mu.$$

Specifically

$$y_- = \log(1-\mu)\cdot r_-(\mu) + C_-,$$
$$y_+ = \log(1+\mu)\cdot r_+(\mu) + C_+,$$

where r_- and r_+ are continous functions in I. The Greens' function G has the following form:

$$(23)\qquad G^*(\mu,\hat{\mu}) = \begin{cases} r_-(\mu)\log(1-\mu) + r_+(\hat{\mu})\cdot\log(1+\hat{\mu}) + C, & \mu \leqq \hat{\mu} \\ r_+(\mu)\log(1+\mu) + r_-(\hat{\mu})\cdot\log(1-\hat{\mu}) + C, & \mu \geqq \hat{\mu}. \end{cases}$$

The square-integrability of G* follows from these expressions, thus completing the proof.

CASE 5

This case ($|f| = 1$, $\ell = 0$) was already considered by Solberg (13). It is degenerate inasmuch as the differential equation has no singularities in the interval I. We have to require as boundary conditions:

$$(24)\qquad \frac{d\theta}{d\mu}(\pm 1) = 0.$$

The eigenvalues are then $\beta_n = \left(\frac{n\Pi}{2}\right)^2$ and the eigenfunctions:

$$(25)\qquad \theta_n = \begin{cases} \sin(\sqrt{\beta_n}\,\mu) & n = 1, 3, 5, \ldots \\ \cos(\sqrt{\beta_n}\,\mu) & n = 0, 2, 4, 6, \ldots \end{cases}$$

It is known that this system too is complete.

Summarizing, one may say: the differential equation of the Hough functions has for each given parameter $\ell \geqq 0$ and $|f| > 0$ simple real eigenvalues β, which have nowhere a finite point of accumulation. For $|f| > 1$, it is always true that $\beta \geqq 0$. The corresponding eigenfunctions constitute a complete orthogonal system in the Hilbert space formed by the functions which are square-integrable in the interval $-1 \leqq \mu \leqq +1$.

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