A New Conservation Theorem of Hydrodynamics

by

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I. Introduction

Let $\mathbf{v}$ denote the velocity vector, $\boldsymbol{\xi} = \nabla \times \mathbf{v}$ the vorticity vector, ∇W the gradient of the action $W = \int_0^t L\,dt$ (Hamilton's Principle function) with the Lagrangian function $L = E_{\mathrm{kin}} - E_{\mathrm{pot}}$ (per unit mass), σ the specific volume of a compressible fluid moving in a potential field Φ, for which a piezotropic relationship, $\rho = \rho(p)$ ($\rho = 1/\sigma$ is the density and p is the pressure), exists for all particles. We prove below that the following conservation theorem holds:

$$\frac{d}{dt}\left\{\sigma\boldsymbol{\xi}\cdot(\mathbf{v} - \nabla W)\right\} = 0, \tag{1}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$$

is the Eulerian operator.

Since $W = 0$ over the entire flow field at time $t = 0$, $(\nabla W)_{t=0} = 0$. Then, the integral form

$$\sigma\boldsymbol{\xi}\cdot(\mathbf{v} - \nabla W) = \overset{\circ}{\sigma}\left(\overset{\circ}{\xi}_x \overset{\circ}{v}_x + \overset{\circ}{\xi}_y \overset{\circ}{v}_y + \overset{\circ}{\xi}_z \overset{\circ}{v}_z\right) \tag{2}$$

expresses the conservation theorem (1), where the scalar product $\sigma\boldsymbol{\xi}\cdot(\mathbf{v} - \nabla W)$ along any trajectory possesses a constant value, i.e., the scalar product

$$(\sigma\boldsymbol{\xi}\cdot\mathbf{v})_{t=0} = \overset{\circ}{\sigma}\left(\overset{\circ}{\xi}_x \overset{\circ}{v}_x + \overset{\circ}{\xi}_y \overset{\circ}{v}_y + \overset{\circ}{\xi}_z \overset{\circ}{v}_z\right) \tag{3}$$

for the given trajectory at the initial time, $t = 0$, and the initial point,

$$x_{t=0} = a, \qquad y_{t=0} = b, \qquad z_{t=0} = c, \tag{4}$$

(x, y, z are orthogonal cartesian coordinates and a, b, c are Lagrangian coordinates).

II. Proof of the Conservation Theorem

(a) Inertial System

On the basis of the Lagrangian hydrodynamical equations in an inertial system,

$$\begin{aligned}
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} &= -\frac{\partial}{\partial a}\left(\Phi + \int\frac{dp}{\rho}\right),\\
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b} &= -\frac{\partial}{\partial b}\left(\Phi + \int\frac{dp}{\rho}\right),\\
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial c} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial c} &= -\frac{\partial}{\partial c}\left(\Phi + \int\frac{dp}{\rho}\right),
\end{aligned} \tag{5}$$

we form their first integrals (Weber's transformation [Ref. (1–5)]):

$$\begin{aligned}\frac{\partial x}{\partial t}\frac{\partial x}{\partial a}+\frac{\partial y}{\partial t}\frac{\partial y}{\partial a}+\frac{\partial z}{\partial t}\frac{\partial z}{\partial a}&=\overset{\circ}{v}_x+\frac{\partial W}{\partial a},\\ \frac{\partial x}{\partial t}\frac{\partial x}{\partial b}+\frac{\partial y}{\partial t}\frac{\partial y}{\partial b}+\frac{\partial z}{\partial t}\frac{\partial z}{\partial b}&=\overset{\circ}{v}_y+\frac{\partial W}{\partial b},\\ \frac{\partial x}{\partial t}\frac{\partial x}{\partial c}+\frac{\partial y}{\partial t}\frac{\partial y}{\partial c}+\frac{\partial z}{\partial t}\frac{\partial z}{\partial c}&=\overset{\circ}{v}_z+\frac{\partial W}{\partial c},\end{aligned} \tag{6}$$

where $\overset{\circ}{v}_x, \overset{\circ}{v}_y, \overset{\circ}{v}_z$ are the wind components at $t = 0$ and

$$W=\int_0^t\left\{\frac{1}{2}v^2-\left(\Phi+\int\frac{dp}{\rho}\right)\right\}dt \tag{7}$$

is Hamilton's principle function (action integral) with

$$E_{\rm kin}=\frac{1}{2}v^2=\frac{1}{2}\mathbf{v}\cdot\mathbf{v} \tag{8}$$

$$E_{\rm pot}=\Phi+\int\frac{dp}{\rho}. \tag{9}$$

The integration over time in (7) is in the "individual" sense, i.e., along the trajectory; each point along the trajectory is assigned a unique time value.

Noting

$$v_x=\frac{\partial x}{\partial t},\qquad v_y=\frac{\partial y}{\partial t},\qquad v_z=\frac{\partial z}{\partial t},$$

and by multiplying Weber's equations (6) by $\partial a/\partial x$, $\partial b/\partial x$, etc., adding appropriate terms, and transforming to Eulerian variables (x, y, z, t), the expressions

$$\begin{aligned}v_x&=\overset{\circ}{v}_x\frac{\partial a}{\partial x}+\overset{\circ}{v}_y\frac{\partial b}{\partial x}+\overset{\circ}{v}_z\frac{\partial c}{\partial x}+\frac{\partial W}{\partial x},\\ v_y&=\overset{\circ}{v}_x\frac{\partial a}{\partial y}+\overset{\circ}{v}_y\frac{\partial b}{\partial y}+\overset{\circ}{v}_z\frac{\partial c}{\partial y}+\frac{\partial W}{\partial y},\\ v_z&=\overset{\circ}{v}_x\frac{\partial a}{\partial z}+\overset{\circ}{v}_y\frac{\partial b}{\partial z}+\overset{\circ}{v}_z\frac{\partial c}{\partial z}+\frac{\partial W}{\partial z},\end{aligned} \tag{10}$$

may be obtained. The equivalent vector form is

$$\mathbf{v}-\nabla W=\overset{\circ}{v}_x\nabla a+\overset{\circ}{v}_y\nabla b+\overset{\circ}{v}_z\nabla c. \tag{11}$$

It follows, through application of the curl operation, that

$$\nabla\times\mathbf{v}=\boldsymbol{\xi}=\nabla\overset{\circ}{v}_x\times\nabla a+\nabla\overset{\circ}{v}_y\times\nabla b+\nabla\overset{\circ}{v}_z\times\nabla c \tag{12}$$

(with $\times$ denoting vector multiplication). Substitution of

$$\begin{aligned}\nabla\overset{\circ}{v}_x&=\frac{\partial\overset{\circ}{v}_x}{\partial a}\nabla a+\frac{\partial\overset{\circ}{v}_x}{\partial b}\nabla b+\frac{\partial\overset{\circ}{v}_x}{\partial c}\nabla c,\\ \nabla\overset{\circ}{v}_y&=\frac{\partial\overset{\circ}{v}_y}{\partial a}\nabla a+\frac{\partial\overset{\circ}{v}_y}{\partial b}\nabla b+\frac{\partial\overset{\circ}{v}_y}{\partial c}\nabla c,\\ \nabla\overset{\circ}{v}_z&=\frac{\partial\overset{\circ}{v}_z}{\partial a}\nabla a+\frac{\partial\overset{\circ}{v}_z}{\partial b}\nabla b+\frac{\partial\overset{\circ}{v}_z}{\partial c}\nabla c,\end{aligned} \tag{13}$$

into (12) results in

$$\boldsymbol{\xi} = \overset{\circ}{\xi}_x \nabla b \times \nabla c + \overset{\circ}{\xi}_y \nabla c \times \nabla a + \overset{\circ}{\xi}_z \nabla a \times \nabla b, \tag{14}$$

where

$$\begin{aligned} \overset{\circ}{\xi}_x &= \frac{\partial \overset{\circ}{v}_z}{\partial b} - \frac{\partial \overset{\circ}{v}_y}{\partial c}, \\ \overset{\circ}{\xi}_y &= \frac{\partial \overset{\circ}{v}_x}{\partial c} - \frac{\partial \overset{\circ}{v}_z}{\partial a}, \\ \overset{\circ}{\xi}_z &= \frac{\partial \overset{\circ}{v}_y}{\partial a} - \frac{\partial \overset{\circ}{v}_x}{\partial b}. \end{aligned} \tag{15}$$

It is noted here that the Lagrangian form is identical to the hydrodynamic vorticity equations as the scalar products

$$\begin{aligned} \boldsymbol{\xi} \cdot \nabla a &= \overset{\circ}{\xi}_x \nabla a \times \nabla b \cdot \nabla c = \overset{\circ}{\xi}_x \frac{\partial(a,b,c)}{\partial(x,y,z)}, \\ \boldsymbol{\xi} \cdot \nabla b &= \overset{\circ}{\xi}_y \nabla a \times \nabla b \cdot \nabla c = \overset{\circ}{\xi}_y \frac{\partial(a,b,c)}{\partial(x,y,z)}, \\ \boldsymbol{\xi} \cdot \nabla c &= \overset{\circ}{\xi}_z \nabla a \times \nabla b \cdot \nabla c = \overset{\circ}{\xi}_z \frac{\partial(a,b,c)}{\partial(x,y,z)}, \end{aligned}$$

which follow from (14), and the functional determinant relations

$$\begin{aligned} \frac{\partial a}{\partial x} &= \frac{\partial(y,z)}{\partial(b,c)} \frac{\partial(a,b,c)}{\partial(x,y,z)}, \\ \frac{\partial a}{\partial y} &= \frac{\partial(z,x)}{\partial(b,c)} \frac{\partial(a,b,c)}{\partial(x,y,z)}, \\ \frac{\partial a}{\partial z} &= \frac{\partial(x,y)}{\partial(b,c)} \frac{\partial(a,b,c)}{\partial(x,y,z)}, \end{aligned}$$

etc., lead to the Cauchy integrals [Ref. (6)]

$$\xi_x \frac{\partial(y,z)}{\partial(b,c)} + \xi_y \frac{\partial(z,x)}{\partial(b,c)} + \xi_z \frac{\partial(x,y)}{\partial(b,c)} = \overset{\circ}{\xi}_x$$

etc. Scalar multiplication of (11) and (14) results in:

$$\boldsymbol{\xi} \cdot (\mathbf{v} - \nabla W) = \left(\overset{\circ}{\xi}_x \overset{\circ}{v}_x + \overset{\circ}{\xi}_y \overset{\circ}{v}_y + \overset{\circ}{\xi}_z \overset{\circ}{v}_z \right) \nabla a \times \nabla b \cdot \nabla c, \tag{16}$$

and by means of the Lagrangian form of the continuity equation

$$\nabla a \times \nabla b \cdot \nabla c = \frac{\partial(a,b,c)}{\partial(x,y,z)} = \frac{\rho}{\overset{\circ}{\rho}} = \frac{\overset{\circ}{\sigma}}{\sigma}, \tag{17}$$

(16) then becomes

$$\sigma\boldsymbol{\xi}\cdot(\mathbf{v}-\nabla W)=\overset{\circ}{\sigma}\left(\overset{\circ}{\xi}_x\overset{\circ}{v}_x+\overset{\circ}{\xi}_y\overset{\circ}{v}_y+\overset{\circ}{\xi}_z\overset{\circ}{v}_z\right), \tag{18}$$

which simultaneously proves equations (1) and (2), since the time independent right-hand side of (18) contains only functions of the Lagrangian coordinates (a, b, c).

In the special case of two dimensional flow, such as flow in the (x, y) plane,

$$\xi_x=\xi_y=0, \qquad v_z=0, \qquad \text{and} \quad \frac{\partial W}{\partial z}=0$$

for $t \geq 0$, so that both sides of equation (2) equal zero. (Also in regard to the special case of two dimensional flow, see, for example, A. Sommerfeld [Ref. (7)].

(b) Rotating Coordinate Systems

Now let

$$\mathbf{r}=\{x(a,b,c,t), y(a,b,c,t), z(a,b,c,t)\} \tag{19}$$

denote the position vector of the particle (a, b, c) at time t in a rotating Cartesian coordinate system (right-handed system), whose rotation is characterized by the constant rotation vector $\boldsymbol{\omega}$ $(|\boldsymbol{\omega}|=\omega)$. If we denote the absolute wind as

$$\mathbf{v}=\frac{\partial\mathbf{r}}{\partial t}+\boldsymbol{\omega}\times\mathbf{r} \tag{20}$$

and the absolute vorticity as

$$\boldsymbol{\xi}=\nabla\times\mathbf{v}=\nabla\times\frac{\partial\mathbf{r}}{\partial t}+2\boldsymbol{\omega}, \tag{21}$$

then the conservation theorem takes the same form as before:

$$\sigma\boldsymbol{\xi}\cdot(\mathbf{v}-\nabla W)=\overset{\circ}{\sigma}\left(\overset{\circ}{\xi}_x\overset{\circ}{v}_x+\overset{\circ}{\xi}_y\overset{\circ}{v}_y+\overset{\circ}{\xi}_z\overset{\circ}{v}_z\right), \tag{22}$$

where the Lagrange function $W=\int_0^t L\,dt$ [Ref. (8)] is given by

$$L=\frac{1}{2}\left(\frac{\partial\mathbf{r}}{\partial t}\right)^2+\omega D_\omega-\left(\overset{*}{\Phi}+\int\frac{dp}{\rho}\right), \tag{23}$$

where

$$\omega D_\omega=\boldsymbol{\omega}\cdot\left(\mathbf{r}\times\frac{\partial\mathbf{r}}{\partial t}\right), \tag{24}$$

$D_\omega=|\mathbf{D}|\cos(\boldsymbol{\omega},\mathbf{D})$ is the projection of the particle's angular momentum (per unit mass), $\mathbf{D}=\mathbf{r}\times(\partial\mathbf{r}/\partial t)$, onto the rotating system's angular velocity, $\boldsymbol{\omega}$, and $\overset{*}{\Phi}=\Phi-\frac{1}{2}(\boldsymbol{\omega}\times\mathbf{r})$ contains the centrifugal potential $-\frac{1}{2}(\boldsymbol{\omega}\times\mathbf{r})$ ($\overset{*}{\Phi}$ is the potential for "gravity" while Φ is the Newtonian attraction potential).

The proof of equation (22) can be given as follows. The rotating Lagrangian hydrodynamic equations of motion

$$
\begin{aligned}
\frac{\partial^2 \mathbf{r}}{\partial t^2}\cdot\frac{\partial \mathbf{r}}{\partial a} + 2\boldsymbol{\omega}\times\frac{\partial \mathbf{r}}{\partial t}\cdot\frac{\partial \mathbf{r}}{\partial a} &= -\frac{\partial}{\partial a}\left(\overset{*}{\Phi} + \int\frac{dp}{\rho}\right),\\
\frac{\partial^2 \mathbf{r}}{\partial t^2}\cdot\frac{\partial \mathbf{r}}{\partial b} + 2\boldsymbol{\omega}\times\frac{\partial \mathbf{r}}{\partial t}\cdot\frac{\partial \mathbf{r}}{\partial b} &= -\frac{\partial}{\partial b}\left(\overset{*}{\Phi} + \int\frac{dp}{\rho}\right),\\
\frac{\partial^2 \mathbf{r}}{\partial t^2}\cdot\frac{\partial \mathbf{r}}{\partial c} + 2\boldsymbol{\omega}\times\frac{\partial \mathbf{r}}{\partial t}\cdot\frac{\partial \mathbf{r}}{\partial c} &= -\frac{\partial}{\partial c}\left(\overset{*}{\Phi} + \int\frac{dp}{\rho}\right),
\end{aligned}
\tag{25}
$$

because of

$$
\begin{aligned}
\boldsymbol{\omega}\times\frac{\partial \mathbf{r}}{\partial t}\cdot\frac{\partial \mathbf{r}}{\partial a} = \frac{\partial}{\partial a}\left(\boldsymbol{\omega}\times\frac{\partial \mathbf{r}}{\partial t}\cdot\mathbf{r}\right) - \boldsymbol{\omega}\times\frac{\partial^2 \mathbf{r}}{\partial a\partial t}\cdot\mathbf{r}\\
= \frac{\partial}{\partial a}\left(\boldsymbol{\omega}\times\frac{\partial \mathbf{r}}{\partial t}\cdot\mathbf{r}\right) - \left\{\frac{\partial}{\partial t}\left(\boldsymbol{\omega}\times\frac{\partial \mathbf{r}}{\partial a}\cdot\mathbf{r}\right) - \boldsymbol{\omega}\times\frac{\partial \mathbf{r}}{\partial a}\cdot\frac{\partial \mathbf{r}}{\partial t}\right\},
\end{aligned}
\tag{26}
$$

from which it follows that

$$
2\boldsymbol{\omega}\times\frac{\partial \mathbf{r}}{\partial t}\cdot\frac{\partial \mathbf{r}}{\partial a} = -\frac{\partial}{\partial a}\left(\omega D_\omega\right) + \frac{\partial}{\partial t}\left(\boldsymbol{\omega}\times\mathbf{r}\cdot\frac{\partial \mathbf{r}}{\partial a}\right),
\tag{27}
$$

(and appropriate equations for variables b, c) permit the representation

$$
\begin{aligned}
\frac{\partial}{\partial t}\left\{\left(\frac{\partial \mathbf{r}}{\partial t} + \omega\times\mathbf{r}\right)\cdot\frac{\partial \mathbf{r}}{\partial a}\right\} - \frac{\partial}{\partial a}\left\{\frac{1}{2}\left(\frac{\partial \mathbf{r}}{\partial t}\right)^2\right\} &= \frac{\partial}{\partial a}\left\{\omega D_\omega - \left(\overset{*}{\Phi} + \int\frac{dp}{\rho}\right)\right\},\\
\frac{\partial}{\partial t}\left\{\left(\frac{\partial \mathbf{r}}{\partial t} + \omega\times\mathbf{r}\right)\cdot\frac{\partial \mathbf{r}}{\partial b}\right\} - \frac{\partial}{\partial b}\left\{\frac{1}{2}\left(\frac{\partial \mathbf{r}}{\partial t}\right)^2\right\} &= \frac{\partial}{\partial b}\left\{\omega D_\omega - \left(\overset{*}{\Phi} + \int\frac{dp}{\rho}\right)\right\},\\
\frac{\partial}{\partial t}\left\{\left(\frac{\partial \mathbf{r}}{\partial t} + \omega\times\mathbf{r}\right)\cdot\frac{\partial \mathbf{r}}{\partial c}\right\} - \frac{\partial}{\partial c}\left\{\frac{1}{2}\left(\frac{\partial \mathbf{r}}{\partial t}\right)^2\right\} &= \frac{\partial}{\partial c}\left\{\omega D_\omega - \left(\overset{*}{\Phi} + \int\frac{dp}{\rho}\right)\right\}.
\end{aligned}
\tag{28}
$$

Through substitution of the absolute wind (20) and the Lagrange function (23) we can thus reduce the Lagrangian equations for an inertial system to the form

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\mathbf{v}\cdot\frac{\partial \mathbf{r}}{\partial a}\right) &= \frac{\partial L}{\partial a},\\
\frac{\partial}{\partial t}\left(\mathbf{v}\cdot\frac{\partial \mathbf{r}}{\partial b}\right) &= \frac{\partial L}{\partial b},\\
\frac{\partial}{\partial t}\left(\mathbf{v}\cdot\frac{\partial \mathbf{r}}{\partial c}\right) &= \frac{\partial L}{\partial c}.
\end{aligned}
\tag{29}
$$

From (29), in congruence with (6), the first integrals

$$
\begin{aligned}
\mathbf{v}\cdot\frac{\partial \mathbf{r}}{\partial a} &= \overset{\circ}{v}_x + \frac{\partial W}{\partial a},\\
\mathbf{v}\cdot\frac{\partial \mathbf{r}}{\partial b} &= \overset{\circ}{v}_y + \frac{\partial W}{\partial b},\\
\mathbf{v}\cdot\frac{\partial \mathbf{r}}{\partial c} &= \overset{\circ}{v}_z + \frac{\partial W}{\partial c},
\end{aligned}
\tag{30}
$$

follow, and the remaining course of calculations follow as above (Section IIa). One recognizes the identity of the two representations (7) and (23) of the Lagrange function by means of the squared equation (20):

$$\frac{1}{2}v^2 = \frac{1}{2}\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}\left(\frac{\partial \mathbf{r}}{\partial t}\right)^2 + \boldsymbol{\omega}\cdot\left(\mathbf{r}\times\frac{\partial \mathbf{r}}{\partial t}\right) + \frac{1}{2}(\boldsymbol{\omega}\times\mathbf{r})^2, \tag{31}$$

which completes the proof from the Lagrangian hydrodynamic equations (25). It is also possible to prove the new conservation theorem using the Eulerian hydrodynamic equations, where a general formulation of the hydrodynamic vorticity theorem is used [Ertel, Ref. (9)]; we elaborate in another place [Ref. (10)].

III. Simplification for incompressible fluids

For incompressible fluids ($\sigma = \overset{\circ}{\sigma}$), the conservation theorem takes the simpler form

$$\boldsymbol{\xi}\cdot(\mathbf{v} - \nabla W) = \overset{\circ}{\xi}_x \overset{\circ}{v}_x + \overset{\circ}{\xi}_y \overset{\circ}{v}_y + \overset{\circ}{\xi}_z \overset{\circ}{v}_z, \tag{32}$$

where in the integrand of W, i.e., within the Lagrange function L, the simplification $\int dp/\rho = p/\rho$ takes place.

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