THESIS

THE THEORY OF STRATIFIED, QUASI-BALANCED FLOWS ON THE SPHERE

Submitted by

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WE HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER OUR SUPERVISION BY LEVI GLENN SILVERS ENTITLED THE THEORY OF STRATIFIED, QUASI-BALANCED FLOWS ON THE SPHERE BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE.

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ABSTRACT OF THESIS

THE THEORY OF STRATIFIED, QUASI-BALANCED FLOWS ON THE SPHERE

This work presents a dynamical model that depicts fully global flows. This model is stratified and the Coriolis force is applied without approximation. The balance condition originally introduced by Charney and Drazin (1961) is used to provide a relationship between the geopotential anomaly field and the streamfunction. Rather than using the geostrophic wind as the advecting quantity, the rotational part of the wind is used for advection. This approximation remains good as long as the large-scale divergence remains small.

An invertibility principle is developed for this balanced model that allows the streamfunction to be calculated if the pseudo-potential vorticity is known. The streamfunction can then be used to compute the pseudo-potential vorticity at the next time step. In this way a predictive system of equations can be implemented to forecast the development of atmospheric flows.

The Charney-Stern necessary condition for combined baroclinic-barotropic instability is generalized to the spherical domain, and the two-layer model is briefly examined. We also modify the index of refraction given by Matsuno (1970) so as to eliminate its singularities at the poles.

The beginning steps of applying our balanced model to geostrophic turbulence are also presented. The Rhines length is generalized to a stratified spherical framework, and the "Parseval relation" is given for both the total energy and the total potential enstrophy.

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DEDICATION

To the glory of God.

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Chapter 1

INTRODUCTION

Atmospheric motions and processes are physically governed by the Navier-Stokes equation. This equation is highly nonlinear and has not submitted to an analytical solution. As a result, the Navier-Stokes equation has been simplified and approximated in various ways in an attempt to gain some understanding of the physical atmosphere. The standard systems of equations that result from these approximations to the Navier-Stokes equations all have strengths and weaknesses. When examining a specific physical scenario or class of motions, it is necessary to choose the system of equations that is 'best' suited to the particular phenomenon under consideration.

The goal of this research is to develop a quasi-balanced theory describing large-scale, small Rossby number flows in a fully spherical domain including motions both at the equator and between the two hemispheres. The derivation of this particular system is similar to traditional derivations of the quasi-geostrophic equations and is detailed in Chapters 2 and 3. Chapters 4 through 6 examine 3 separate examples of problems that have traditionally been studied on the f-plane or the β-plane. This work further generalizes each of the problems to quasi-balanced flows on the sphere. The ability of the theory presented with this research to concisely generalize each of these problems provides the primary argument for the legitimacy of the quasi-balanced model.

The quasi-balanced set of equations result from selectively applying the appropriate scaling approximations to the quasi-static primitive equations (which are derived from the Navier-Stokes equation) and assuming that hydrostatic balance is maintained. We then implement the balance condition to derive an invertibility principle. This set of equations has been developed as a system that is useful when describing motions for which the length scale is large enough that the Rossby number (as defined in Chapter 2) will be sufficiently small ($Ro << 1$). The Rossby number is the ratio of the geostrophic adjustment time scale to the advective time scale. Thus, for small Rossby numbers, the geostrophic adjustment time scale is faster than the advective time scale to the extent that the motions remain close to geostrophic balance.

Historically, the quasi-geostrophic equations have proven to be extremely useful. Our quasi-balanced model is similar in many ways to the quasi-geostrophic system and is intended to be a helpful tool that provides insight into the mechanisms driving atmospheric flows. To provide some historical background, quasi-geostrophic theory is briefly discussed followed by some of the reasoning that inspired the choice to use the balanced theory in this work.

Geostrophic balance is defined as motion occurring in fluids when the component of the Coriolis force normal to the gravitational force is balanced by the horizontal component of the pressure gradient force. When friction or other additional accelerations are considered, but the normal component of the Coriolis force and the pressure gradient force are still largely balanced, the flow is said to be ‘approximately geostrophic’.

As a beginning point for dynamical analysis, approximating the atmospheric flow to be in geostrophic balance gives a simplified, but still informative and helpful framework through which to view the observed atmospheric motions. From the Navier-Stokes equation there are five scalar prognostic equations governing atmospheric motions. These equations mathematically represent the three components of velocity, the temperature, and the density. When the hydrostatic and geostrophic approximations are introduced into this system, surprisingly, the five prognostic equations reduce to one. Edward Lorenz gives an elucidating physical explanation for this process of simplification (Lorenz, 2006).

The reduction of the dynamical system of equations to one prognostic equation was the most significant single development leading to the reality of numerical weather prediction. Breakthroughs in dynamic meteorology as a result of this development are still occurring. The quasi-geostrophic system allows for physical interpretations of the simplified equations in ways that are not possible with the original primitive equations. It is difficult to fully grasp the significance of what we have learned through geostrophic analysis, in the words of Lorenz, "I personally regard the successful reduction of the dynamic equations to a single prognostic equation by means of the geostrophic relationship, entirely apart from any applicability to numerical weather prediction, as the greatest single achievement of twentieth-century dynamic meteorology."

To cast a given atmospheric system into the framework of quasi-geostrophic theory, the observed flow is divided into two parts: the geostrophic flow and the ageostrophic flow. The geostrophic flow is defined as the flow that is balanced between the coriolis force and the pressure gradient force, while the ageostrophic flow is simply defined as the difference between the observed flow and the geostrophic flow. For quasi-geostrophic theory to be a useful approximation of the atmospheric phenomena being studied, the ageostrophic flow must be significantly smaller in magnitude than the geostrophic flow.

Although the ageostrophic flow is a proportionally small part of the flow, it plays a critical role in the dynamical development of the system. As the flow evolves, it can be shown that the geostrophic advection tends to destroy the thermal wind balance. However, thermal wind balance must be maintained for the required balance conditions of the geostrophic system to be satisfied. Thus for a quasi-geostrophic system to maintain itself, there must be another element which helps to maintain the twin requirements of hydrostatic and geostrophic balance. This is supplied by a secondary circulation which is forced by the geostrophic wind. Thus the ageostrophic wind is a vital component to maintaining the thermal wind balance.

Analysis of flows at a small Rossby number presents us with an interesting choice: should the equations be written and computed in terms of the geopotential height field, or should they be written out in terms of a streamfunction? Both the geopotential field and the streamfunction have been used in previous models, although the geopotential appears to have won the popularity contest. We have chosen to use a streamfunction primarily for two reasons. First, the computation of velocity through the geopotential leads to a singular point at the equator. This does not happen when the streamfunction is used. Because of this difficulty with the geopotential field, the quasi-geostrophic system is usually applied only to the midlatitudes. The goal of this research is to apply our balanced theory globally.

The second reason the streamfunction was chosen is well illustrated from contrasting forecasts made by Jule Charney and Norman Phillips of a particular storm that occurred on November 24, 1950. Charney attempted to predict the 24-hour cyclogenesis of this storm using a three-layer model that predicted the geopotential field (Charney, 1954). His model failed to correctly predict the strength and location of the storm's development. Phillips (1958) performed integrations of the same storm with a two-layer model that predicted the streamfunction rather than the geopotential field. Phillips' model produced a much better forecast for the developing center of the storm.

Although these two models differed in such fundamental ways as the number of levels represented, close inspection reveals that the difference in the resulting forecast was due primarily to the fact that Charney's model used the geopotential field, while Phillips' model used the streamfunction. Phillips summarized the situation nicely by saying that the question of which field to use, "depends on the structure of the atmospheric flow pattern, and the relative importance of vorticity and temperature advection."

Mathematically, the geopotential field and the streamfunction are proportional to one another with the proportionality factor being the Coriolis parameter. The difference between models using the streamfunction or the geopotential arises through the computation of each quantity. One can see through the thickness (or hypsometric) equation the direct dependence of the geopotential height on the temperature in the layer of atmosphere being measured. The winds are then calculated from the resultant values of the geopotential. In contrast, the streamfunction field gives a direct representation of the winds.

For Charney's model, he essentially took the temperature reading at various locations, and then calculated the corresponding geopotential for each location. Numerical integrations were then computed based on the resultant geopotential fields. On the other hand, Phillips took the wind measurements, used them to compute the corresponding streamfunction, and used the resultant fields in his model. This reveals that although the geopotential and streamfunction fields are mathematically proportional to each other, computationally they are very different; the former depends on temperature, while the latter depends on wind.

It appears that for the particular storm of November 24, 1950, the advection of vorticity was more critical than the advection of temperature to accurately predict the developing storm. Charney's model overestimated the wind and thus forecast advection of the vorticity too far north. The differences between these two methods of forecasting can be summarized as follows. For an analysis relying on the geopotential field, the initial temperature field will be exact but the wind field will only be good insofar as the geostrophic approximation represents the wind at that particular time. In contrast, the streamfunction gives an exact wind field that is not dependent on the geostrophic approximation, while the temperature field will only be approximate. When forecasting the development of a particular system it is critical to determine which of these two fields is the most representative of the dynamics.

Classical quasi-geostrophic models use the geostrophic wind for advection, while the model presented in this research uses the rotational part of the wind for the advection. This is a critical difference of our model and provides a distinct advantage. The geostrophic wind has restrictions on it due to the dependence of the Coriolis parameter on latitude, while the rotational part of the flow has no such restrictions. Our system presented here is then not strictly a quasi-geostrophic model, but a quasi-balanced model.

The mathematical assumptions and governing equations are presented in Chapters 2 and 3. The rest of the thesis re-examines a few classical problems that have historically been studied with quasi-geostrophic theory. This allows both the similarities and differences between previous versions of quasi-geostrophic theory and this new system to be seen.

The Charney-Stern necessary condition for combined barotropic-baroclinic instability is generalized in Chapter 4. This chapter includes two derivations of the necessary condition for combined barotropic-baroclinic instability; one involving a time dependent solution, the other depending on a meridional displacement of a fluid parcel. The chapter concludes by briefly examining the two-layer model.

Chapter 5 looks at the vertical propagation of Rossby waves. The pseudo-index of refraction that was defined by Matsuno (1970) is altered and the implied results stated. The equations leading to geostrophic turbulence are presented in Chapter 6, with a discussion on potential future work on global geostrophic turbulence. The thesis is completed with the conclusions in Chapter 7 and a discussion of the strengths and weaknesses of the spherical quasi-balanced theory.

These three topics, the Charney-Stern theorem, vertical propagation of planetary waves, and geostrophic turbulence, have not been examined with the intention of discovering previously unknown scientific principles. Rather, they provide a means of verifying the validity of this model. In the following chapters it is shown that the quasi-balanced model allows for the successful generalizations to the sphere of each of the topics examined. In addition, it is shown that the quasi-balanced model contains a reasonable total energy equation. Each of these successful analytical developments implies that the quasi-balanced model is able to correctly approximate the large-scale motions of the atmosphere we are interested in studying. The next step will be to numerically develop this model and utilize its spherical, fully-global domain. Thus, this work provides a springboard from which future research can be undertaken.

Chapter 2

DERIVATION OF QUASI-BALANCED THEORY ON THE SPHERE

2.1 Governing Equations

Consider the large-scale, quasi-static, adiabatic motions of a stratified, compressible atmosphere on the sphere. Using longitude λ and latitude ϕ as the horizontal coordinates and potential temperature $\theta = T(p_0/p)^\kappa$ as the vertical coordinate, we can write the quasi-static primitive equations as

$$\frac{Du}{Dt} - \left(2\Omega\sin\phi + \frac{u\tan\phi}{a}\right)v + \left(\frac{\partial M}{a\cos\phi\partial\lambda}\right)_\theta = 0, \tag{2.1}$$

$$\frac{Dv}{Dt} + \left(2\Omega\sin\phi + \frac{u\tan\phi}{a}\right)u + \left(\frac{\partial M}{a\partial\phi}\right)_\theta = 0, \tag{2.2}$$

$$\frac{\partial M}{\partial\theta} = \Pi, \tag{2.3}$$

$$\frac{D\sigma}{Dt} + \sigma\left[\left(\frac{\partial u}{a\cos\phi\partial\lambda}\right)_\theta + \left(\frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi}\right)_\theta\right] = 0, \tag{2.4}$$

where u is the eastward component of velocity, v the northward component, $\kappa = R/c_p$, R the gas constant of dry air, c_p the specific heat of dry air at constant pressure, $M = c_pT+\Phi$ the Montgomery potential, Φ the geopotential, T the temperature, $\Pi = c_p(p/p_0)^\kappa$ the Exner function, p_0 the global mean sea level pressure, $\sigma = -g^{-1}(\partial p/\partial\theta)$ the pseudo-density, Ω the earth's rotation rate, a the earth's radius, and

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_\theta + u\left(\frac{\partial}{a\cos\phi\partial\lambda}\right)_\theta + v\left(\frac{\partial}{a\partial\phi}\right)_\theta \tag{2.5}$$

the material derivative. Equations (2.1) and (2.2) are the momentum equations for the eastward and northward components of velocity. The hydrostatic relationship is expressed in (2.3), while the continuity equation is given in (2.4) and ensures the conservation of mass in our system. The vorticity equation, derived from cross differentiating (2.1) and (2.2) , is

$$\frac{D\zeta_\theta}{Dt} + \zeta_\theta \left[\left(\frac{\partial u}{a\cos\phi\partial\lambda} \right)_\theta + \left(\frac{\partial (v\cos\phi)}{a\cos\phi\partial\phi} \right)_\theta \right] = 0, \tag{2.6}$$

where

$$\zeta_\theta = 2\Omega\sin\phi + \left(\frac{\partial v}{a\cos\phi\partial\lambda} \right)_\theta - \left(\frac{\partial (u\cos\phi)}{a\cos\phi\partial\phi} \right)_\theta \tag{2.7}$$

is the isentropic absolute vorticity. The potential vorticity equation, derived by combining (2.4) and (2.6), is

$$\frac{DP}{Dt} = 0, \tag{2.8}$$

where $P = \zeta_\theta/\sigma$ is the potential vorticity.

We shall now derive a global quasi-balanced theory by direct approximation of (2.8). The final form of the theory is most conveniently expressed in a pressure-type coordinate, and here we shall use the pseudo-height $z = (c_p\theta_0/g)[1 - (p/p_0)^\kappa]$ for our vertical coordinate, where θ_0 is a constant reference potential temperature and p_0 is the previously defined global mean sea level pressure. The pseudo-height coordinates are convenient because the thermal wind relationship takes a very simple form. In addition, the thermodynamic equation and the hydrostatic equation are both formulated in terms of the potential temperature. This is not the case for most of the other standard coordinate systems. The transformation between the isentropic coordinates and the pseudo-height coordinates is accomplished using the transformation relations

$$\left(\frac{\partial}{\partial\lambda} \right)_\theta = \left(\frac{\partial}{\partial\lambda} \right)_z - \left(\frac{\partial\theta}{\partial\lambda} \right)_z \frac{\partial z}{\partial\theta} \frac{\partial}{\partial z}, \tag{2.9}$$

$$\left(\frac{\partial}{\partial\phi} \right)_\theta = \left(\frac{\partial}{\partial\phi} \right)_z - \left(\frac{\partial\theta}{\partial\phi} \right)_z \frac{\partial z}{\partial\theta} \frac{\partial}{\partial z}, \tag{2.10}$$

$$\left(\frac{\partial}{\partial t} \right)_\theta = \left(\frac{\partial}{\partial t} \right)_z . \tag{2.11}$$

The pseudo-density in pseudo-height coordinates becomes

$$\rho(z) = \rho_0[1 - gz/(c_p\theta_0)]^{\frac{1-\kappa}{\kappa}} . \tag{2.12}$$

Potential vorticity can thus be written as

$$P = \frac{1}{\rho}\left\{-\frac{\partial v}{\partial z}\left(\frac{\partial\theta}{a\cos\phi\partial\lambda}\right)_z + \frac{\partial u}{\partial z}\left(\frac{\partial\theta}{a\partial\phi}\right)_z + \left[2\Omega\sin\phi + \left(\frac{\partial v}{a\cos\phi\partial\lambda}\right)_z - \left(\frac{\partial(u\cos\phi)}{a\cos\phi\partial\phi}\right)_z\right]\frac{\partial\theta}{\partial z}\right\}. \tag{2.13}$$

In addition, using (2.9)–(2.11) we can show that

$$\frac{D}{Dt} = \frac{\mathcal{D}_z}{\mathcal{D}t} - \frac{\mathcal{D}_z\theta}{\mathcal{D}t}\frac{\partial z}{\partial\theta}\frac{\partial}{\partial z}, \tag{2.14}$$

where

$$\frac{\mathcal{D}_z}{\mathcal{D}t} = \left(\frac{\partial}{\partial t}\right)_z + u\left(\frac{\partial}{a\cos\phi\partial\lambda}\right)_z + v\left(\frac{\partial}{a\partial\phi}\right)_z. \tag{2.15}$$

We now define the potential temperature anomaly (or deviation) by $\theta_d = \theta - \theta_s(z)$ and the geopotential anomaly by $\Phi_d = \Phi - \Phi_s(z)$, where $\theta_s(z)$ and $\Phi_s(z)$ are standard atmosphere potential temperature and geopotential profiles, with $\theta_s(0) = \theta_0$. The temperature and geopotential fields are related hydrostatically, both for the reference state and for the anomaly, i.e., $\partial\Phi_s/\partial z = (g/\theta_0)\theta_s$ and $\partial\Phi_d/\partial z = (g/\theta_0)\theta_d$. The reference potential temperature field is assumed to be monotonic so that $N^2(z) = (g/\theta_0)(d\theta_s/dz)$, the square of the Brunt-Väisälä frequency, is positive for all z. We also define the potential vorticity anomaly by $P_d = P - P_s(\phi, z)$, where $P_s(\phi, z) = \rho^{-1}2\Omega\sin\phi\,(d\theta_s/dz)$ is the potential vorticity of a resting atmosphere with the horizontally uniform potential temperature field $\theta_s(z)$. Using these definitions and the material derivative relation (2.14), we can write (2.8) as

$$\frac{\mathcal{D}_z P_d}{\mathcal{D}t} + v\left(\frac{\partial P_s}{a\partial\phi}\right)_z - \frac{\partial z}{\partial\theta}\frac{\partial P}{\partial z}\frac{\mathcal{D}_z\theta_d}{\mathcal{D}t} = 0. \tag{2.16}$$

It is important to note that (2.16) follows exactly from (2.8). Essentially all we have done is to convert (2.8) to a new coordinate system.

We now make the quasi-balanced approximation to (2.16). To do this, consider motions that have the characteristic horizontal scale L, the characteristic vertical scale D, and the characteristic velocity U. Define the Rossby and Richardson numbers by $\mathrm{Ro} = U/(2\Omega L)$ and $\mathrm{Ri} = N_0^2/(U/D)^2$ respectively, where N_0 is the constant mean value of $N(z)$. We now restrict the discussion to motions characterized by $\mathrm{Ro} \ll 1$ and $\mathrm{Ro}^2\mathrm{Ri} \sim 1$. Under this restriction, our approximation to

(2.16) is

$$\frac{\mathcal{D}}{\mathcal{D}t}\left(P_d - \frac{1}{N^2}\frac{\partial P_s}{\partial z}\frac{g}{\theta_0}\theta_d\right) + v_\psi \left(\frac{\partial P_s}{a\partial\phi}\right)_z = 0 \tag{2.17}$$

where

$$\frac{\mathcal{D}}{\mathcal{D}t} = \left(\frac{\partial}{\partial t}\right)_z + u_\psi\left(\frac{\partial}{a\cos\phi\partial\lambda}\right)_z + v_\psi\left(\frac{\partial}{a\partial\phi}\right)_z \tag{2.18}$$

is an approximation to $\mathcal{D}_z/\mathcal{D}t$ based on the rotational wind components

$$u_\psi = -\left(\frac{\partial\psi}{a\partial\phi}\right)_z, \qquad v_\psi = \left(\frac{\partial\psi}{a\cos\phi\partial\lambda}\right)_z, \tag{2.19}$$

and P_d is approximated by

$$\begin{aligned} P_d &= \frac{1}{\rho}2\Omega\frac{d\theta_s}{dz}\left[\frac{\nabla_z^2\psi}{2\Omega} + \frac{\mu}{N^2}\frac{g}{\theta_0}\frac{\partial\theta_d}{\partial z} + \frac{\nabla_z^2\psi}{2\Omega}\frac{g}{N^2\theta_0}\frac{\partial\theta_d}{\partial z} + \frac{g}{2\Omega N^2\theta_0}\left(\frac{\partial u}{\partial z}\frac{\partial\theta_d}{a\partial\phi} - \frac{\partial v}{\partial z}\frac{\partial\theta_d}{a\cos\phi\partial\lambda}\right)\right] \\ &\approx \frac{1}{\rho}\frac{d\theta_s}{dz}\left(\nabla_z^2\psi + \frac{2\Omega\mu}{N^2}\frac{\partial^2\Phi_d}{\partial z^2}\right). \end{aligned} \tag{2.20}$$

Note that the first line of (2.20) follows exactly from (2.13), and that the four terms in brackets have the respective magnitudes Ro, (Ro Ri)$^{-1}$, Ri^{-1}, Ri^{-1}. For motions characterized by $\mathrm{Ro} \ll 1$ and $\mathrm{Ro}^2\mathrm{Ri} \sim 1$ the last two terms can be neglected, so that the P_d in (2.17) is approximated by the second line in (2.20). Then, defining

$$q = \frac{g\rho}{\theta_0 N^2}P_d + 2\Omega\mu\frac{d}{\rho dz}\left(\frac{\rho}{N^2}\right)\frac{\partial\Phi_d}{\partial z} = \nabla_z^2\psi + 2\Omega\mu\frac{\partial}{\rho\partial z}\left(\frac{\rho}{N^2}\frac{\partial\Phi_d}{\partial z}\right), \tag{2.21}$$

we can write (2.17) as

$$\frac{\mathcal{D}q}{\mathcal{D}t} + \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} = 0. \tag{2.22}$$

This is our spherical, quasi-balanced, pseudo potential vorticity equation. It should be noted that, because of the second term after the first equality in (2.21), the quasi-geostrophic pseudo potential vorticity anomaly q is not to be regarded as an approximation of the potential vorticity anomaly P_d.

Concerning the boundary conditions for (2.21), we here assume that the potential temperature anomaly vanishes at the top boundary, which leads via the hydrostatic equation to the boundary condition given in the first half of (2.23). At the lower isobaric surface we assume that parcels do not cross physical height surfaces, i.e., $D\Phi/Dt = 0$ at $z = 0$. We approximate this by

$\mathcal{D}\Phi_d/\mathcal{D}t + gw = 0$. At $z = 0$ the quasi-geostrophic form of the thermodynamic equation is $\mathcal{D}/\mathcal{D}t(\partial\Phi_d/\partial z) + N^2 w = 0$. Eliminating w between these last two equations, we obtain the condition $\mathcal{D}/\mathcal{D}t[(\partial\Phi_d/\partial z) - (N^2/g)\Phi_d] = 0$ at $z = 0$. We now multiply this last equation by $2\Omega\mu$ and consider this factor to be slowly varying to obtain the lower boundary condition. Utilizing the relationship $\Phi_d = 2\Omega\mu\psi$ which will be justified in the next chapter, the boundary conditions at the top and bottom of the domain are then

$$\frac{\partial\psi}{\partial z} = 0 \quad \text{at } z = z_T, \qquad \frac{\mathcal{D}B}{\mathcal{D}t} = 0 \quad \text{at } z = 0. \tag{2.23}$$

In summary, the governing prognostic equation for the interior of the domain is

$$\frac{\partial q}{\partial t} + \frac{1}{a^2}\frac{\partial(\psi, q)}{\partial(\lambda, \mu)} + \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} = 0, \tag{2.24}$$

where $\frac{\partial(\psi,q)}{\partial(\lambda,\mu)}$ is the Jacobian operator. The governing equation for the lower boundary is

$$\frac{\partial B}{\partial t} + \frac{1}{a^2}\frac{\partial(\psi, B)}{\partial(\lambda, \mu)} = 0, \tag{2.25}$$

where

$$B = 4\Omega^2\mu^2\left(\frac{\partial\psi}{\partial z} - \frac{N^2}{g}\psi\right) \quad \text{at } z = 0. \tag{2.26}$$

2.2 The Total Energy Principle

We would now like to know if this balanced theory conserves energy. It is not guaranteed after approximating a given set of equations that they will obey a reasonable energy principle. The following derivation however shows spherical quasi-balanced does conserve energy over the global domain.

We derived two governing equations for this system; one for the interior of the fluid, and one for the lower boundary. To obtain an energy principle, we first multiply these equations by ψ and then integrate over the entire mass for the interior, and over the surface of the sphere for the lower boundary. Thus from (2.24) for the interior we have

$$\int_0^{z_T}\int_{-1}^{1}\int_0^{2\pi}\left[\psi\frac{\partial q}{\partial t} + \frac{\psi}{a^2}\frac{\partial(\psi, q)}{\partial(\lambda, \mu)} + \frac{2\Omega}{a^2}\psi\frac{\partial\psi}{\partial\lambda}\right]\rho\, a^2\, d\lambda\, d\mu\, dz = 0, \tag{2.27}$$

and from (2.25) for the lower surface we have

$$\int_{-1}^{1}\int_{0}^{2\pi}\left[\psi\frac{\partial B}{\partial t}+\frac{\psi}{a^2}\frac{\partial(\psi,B)}{\partial(\lambda,\mu)}\right]\rho\, a^2\, d\lambda\, d\mu=0. \tag{2.28}$$

Equation (2.27) can also be written as

$$\int_{0}^{z_T}\int_{-\pi/2}^{\pi/2}\int_{0}^{2\pi}\left[\psi\frac{\partial q}{\partial t}+\frac{\partial\left(q\psi v_\psi\cos\phi\right)}{a\cos\phi\partial\phi}+\frac{\partial\left(q\psi u_\psi\right)}{a\cos\phi\partial\lambda}+\frac{2\Omega}{a^2}\frac{\partial\psi^2}{2\partial\lambda}\right]\rho\, a^2\,\cos\phi\, d\lambda\, d\phi\, dz=0. \tag{2.29}$$

The third and fourth terms in brackets integrate to zero because they are in the form of perfect differentials, which when integrated around a closed loop equal zero. The second term integrates to zero because the cosine function evaluated at the given limits equals zero. All that remains of the above integral is the first term in brackets.

Equation (2.28) can be evaluated in an analogous way, showing that only the term involving the time derivative is nonzero. We can then combine these two results to obtain

$$\int_{0}^{z_T}\int_{-1}^{1}\int_{0}^{2\pi}\psi\frac{\partial q}{\partial t}\,\rho\, d\lambda\, d\mu\, dz+\int_{-1}^{1}\int_{0}^{2\pi}\psi\frac{\partial B}{\partial t}\rho\, d\lambda\, d\mu=0. \tag{2.30}$$

Using (2.21) and (2.25), we obtain the total energy conservation principle

$$\frac{dE}{dt}=0, \tag{2.31}$$

where

$$E=\int_{0}^{z_T}\int_{-1}^{1}\int_{0}^{2\pi}\frac{1}{2}\left[\nabla\psi\cdot\nabla\psi+\frac{4\Omega^2\mu^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2\right]\rho\, d\lambda\, d\mu\, dz+\int_{-1}^{1}\int_{0}^{2\pi}\frac{2\Omega^2\mu^2}{g}\psi_0^2\rho\, d\lambda\, d\mu \tag{2.32}$$

is the total energy. It should be mentioned that the exact form of (2.21) used above in (2.30) was actually the form of q derived in the next chapter. However, this form of q is derived directly from (2.21). The first term to the right of the equals sign in the above equation represents the kinetic energy and the available potential energy due to temperature perturbations in the interior of the fluid. The second term on the right of the equals sign represents the available potential energy due to the height differences on the lower isobaric surface. The system presented in this chapter thus conserves energy, as it should.

Chapter 3

THE INVERTIBILITY PRINCIPLE

One of the primary ways of making this theory useful to dynamic meteorology is to develop a way in which the governing equation can be used to predict the evolving fields. A common way of doing this is to use the conservation of potential vorticity as the predictive equation, and then to develop an 'invertibility principle' as a way to diagnose the balanced wind and mass fields from the potential vorticity field. This chapter develops the relevant invertibility principle.

Recall from the previous chapter

$$q = \nabla_z^2 \psi + 2\Omega\mu \, \frac{\partial}{\rho \partial z} \left(\frac{\rho}{N^2} \frac{\partial \Phi_d}{\partial z} \right) . \tag{3.1}$$

In the above equation we find a relationship between the pseudo-potential vorticity anomaly (q), the streamfunction (ψ), and the geopotential anomaly (Φ_d). In order to convert (3.1) into an invertibility principle we must relate the geopotential anomaly Φ_d to the streamfunction ψ. Following the argument of Charney and Drazin (1961), and Charney and Stern (1962) we now assume that ψ and Φ_d are related by the linear balance condition $\nabla_z \cdot (2\Omega\mu \nabla_z \psi) = \nabla_z^2 \Phi_d$, and that $2\Omega\mu$ can be considered a slowly varying function, so that the balance condition can be simplified to $\nabla_z^2(\Phi_d - 2\Omega\mu\psi) = 0$, from which it immediately follows that

$$\Phi_d = 2\Omega\mu\psi. \tag{3.2}$$

If we now substitute (3.2) into (3.1) and obtain

$$q = \nabla^2 \psi + 4\Omega^2 \mu^2 \, \frac{\partial}{\rho \partial z} \left(\frac{\rho}{N^2} \frac{\partial \psi}{\partial z} \right) . \tag{3.3}$$

The subscript z has been dropped from the Laplacian operator in (3.3) for simplicity but it is understood that the horizontal Laplacian is being represented. Equation (3.3) combined with the boundary conditions defined in the previous chapter constitute an invertibility relation from which the streamfunction ψ can be calculated if q is known in the interior and B is known on the lower boundary.

We now use a vertical transform to eliminate the vertical derivatives in (3.3). Define $\psi_\ell(\lambda, \mu, t)$ as the vertical integral transform of $\psi(\lambda, \mu, z, t)$ by the relation

$$\psi_\ell(\lambda, \mu, t) = \int_0^{Z_T} \psi(\lambda, \mu, z, t) V_\ell(z) e^{-z/2} \, dz, \tag{3.4}$$

with the inverse relation being

$$\psi(\lambda, \mu, z, t) = \sum_{\ell=0}^{\infty} \psi_\ell(\lambda, \mu, t) V_\ell(z) e^{z/2} \tag{3.5}$$

where $e^{-z/2}$ is the weight and $V_\ell(z)$ is the kernel of the transform and the subscripts ℓ designate vertical modes. We require the kernel of the transform to satisfy a Sturm-Liouville eigenproblem, which implies that the eigenfunctions are orthogonal. An analogous definition is made for $q_\ell(\lambda, \mu, t)$, the vertical integral transform of $q(\lambda, \mu, z, t)$.

To vertically transform (3.3), we multiply it by $V_\ell(z)e^{-z/2}$ and then integrate the result from $z = 0$ to $z = z_T$. Thus we obtain

$$\nabla^2 \psi_\ell - \frac{\epsilon_\ell \mu^2}{a^2} \psi_\ell = q_\ell \tag{3.6}$$

where $\epsilon_\ell = 4\Omega^2 a^2/(gh_\ell)$ is Lamb's parameter for each particular vertical mode ℓ. Although we have introduced multiple vertical modes, it can be seen that (3.6) is a simpler equation than (3.3) due to the lack of vertical derivatives. Solutions to (3.6) can be obtained by finding the Green function for this equation. To do this, the horizontal transform of ψ_ℓ and q_ℓ is taken using the following

$$\psi_\ell(\lambda, \mu) = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \psi_{\ell mn} \mathcal{S}_{mn}(\epsilon_\ell; \lambda, \mu) \tag{3.7}$$

and

$$\psi_{\ell mn} = \frac{1}{4\pi} \int_{-1}^{1} \int_0^{2\pi} \psi_\ell(\lambda, \mu) \mathcal{S}^*_{mn}(\epsilon_\ell; \lambda, \mu) \, d\lambda \, d\mu \tag{3.8}$$

where $\mathcal{S}^*$ is the complex conjugate of $\mathcal{S}$ with m as the zonal wavenumber and n as the total wavenumber. An analogous transform relationship for q_ℓ is defined similarly. It proves useful if the kernel of the transform pair above is given by the spheroidal harmonics $\mathcal{S}_{mn}(\epsilon_\ell; \lambda, \mu)$. A concise summary of spheroidal harmonics is given in Abramowitz and Stegun (1965, pages 751-769). A very helpful and extensive analysis is given by Flammer (1957). Substituting (3.7) into (3.6) gives

$$\sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \psi_{\ell mn} \nabla^2 \mathcal{S}_{mn} - \frac{\epsilon_\ell \mu^2}{a^2} \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \psi_{\ell mn} \mathcal{S}_{mn} = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} q_{\ell mn} \mathcal{S}_{mn} \tag{3.9}$$

which can be written as

$$\sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \psi_{\ell mn} \left\{ \nabla^2 \mathcal{S}_{mn} - \frac{\epsilon_\ell \mu^2}{a^2} \mathcal{S}_{mn} \right\} = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} q_{\ell mn} \mathcal{S}_{mn}. \tag{3.10}$$

But it is also known that spheroidal harmonics satisfy

$$\nabla^2 \mathcal{S}_{mn} - \frac{\epsilon_\ell \mu^2}{a^2} \mathcal{S}_{mn} = -\frac{\alpha_{mn}(\epsilon_\ell)}{a^2} \mathcal{S}_{mn}. \tag{3.11}$$

where $-\alpha_{mn}(\epsilon_\ell)/a^2$ is the eigenvalue. This allows us to write (3.10) as

$$\sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \left(-\frac{\alpha_{mn}(\epsilon_\ell)}{a^2} \psi_{\ell mn} - q_{\ell mn} \right) \mathcal{S}_{mn} = 0. \tag{3.12}$$

Thus

$$\psi_{\ell mn} = -\frac{a^2}{\alpha_{mn}} q_{\ell mn}. \tag{3.13}$$

Substituting this into (3.7) gives

$$\psi_\ell(\lambda, \mu) = -\sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \frac{a^2}{\alpha_{mn}(\epsilon_\ell)} q_{\ell mn} \mathcal{S}_{mn}(\epsilon_\ell; \lambda, \mu). \tag{3.14}$$

Plugging in the transform of $q_{\ell mn}$ gives

$$\psi_\ell(\lambda, \mu) = -\sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \frac{a^2}{\alpha_{mn}(\epsilon_\ell)} \left\{ \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} q_\ell(\lambda', \mu') \mathcal{S}^*_{mn}(\epsilon_\ell; \lambda', \mu') \, d\lambda' \, d\mu' \right\} \mathcal{S}_{mn}(\epsilon_\ell; \lambda, \mu), \tag{3.15}$$

where the primed variables indicate 'dummy' variables that are being integrated over. We can bring the summations inside of the integrals if we are careful to keep all variables that are summed over m and n inside of the summations.

$$\psi_\ell(\lambda, \mu) = -\frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} q_\ell(\lambda', \mu') \left\{ \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \frac{a^2}{\alpha_{mn}(\epsilon_\ell)} \mathcal{S}_{mn}(\epsilon_\ell; \lambda, \mu) \mathcal{S}^*_{mn}(\epsilon_\ell; \lambda', \mu') \right\} d\lambda' \, d\mu'. \tag{3.16}$$

Note that the expression inside of the braces is the Green function for (3.6). In other words

$$\psi_\ell(\lambda, \mu) = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} q_\ell(\lambda', \mu') G(\epsilon_\ell; \lambda - \lambda', \mu, \mu') \, d\lambda' \, d\mu' \tag{3.17}$$

where

$$G(\epsilon_\ell; \lambda - \lambda', \mu, \mu') = - \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \frac{a^2}{\alpha_{mn}(\epsilon_\ell)} S_{mn}(\epsilon_\ell; \mu) S^*_{mn}(\epsilon_\ell; \mu') e^{im(\lambda - \lambda')}. \tag{3.18}$$

Both the Green function (3.18), and equation (3.14) provide solutions to (3.6). Provided the eigenvalues can be found without too much trouble usually (3.14) is computationally a better solution. The Green function however is often an important analytical tool that provides us with additional physical understanding.

Chapter 4

BAROTROPIC-BAROCLINIC INSTABILITY

4.1 Introduction

Historically, the theory of hydrodynamic instability has attracted much attention. Significant contributions have been made by Rayleigh (1880), Taylor (1915), Eady (1949), Fjørtoft (1950), Charney (1947), Charney and Stern (1962), and Eliassen (1983), to name just a few. Investigating flows that simultaneously include both horizontal and vertical shear initially proved to be too complex of a problem. As a result, the analysis has often been simplified by looking at flows which contain either horizontal shear (barotropic) or vertical shear (baroclinic). Charney and Stern's paper of 1962 presented the first attempt to tackle the problem of combined barotropic-baroclinic instability by generalizing some of Rayleigh's original results.

The difference between barotropic and baroclinic instabilities can be clarified in the following way. Due to uneven radiative heating of the planet, a temperature gradient between the equator and the poles is constantly generated. The resulting tilted surfaces of constant density provide a source of potential energy. Any process that acts to 'level' these surfaces will release some of this potential energy. A baroclinically unstable flow will draw from this source of energy to strengthen and maintain initial disturbances of the mean flow. In contrast, barotropic instability will draw its strength from the kinetic energy of a system (often related to horizontal shears). Combined baroclinic-barotropic can be present in flows where there is both horizontal and vertical dependence present.

The endeavors of the previously mentioned authors have managed to generalize the theory developed by Rayleigh which analyzed the stability of two-dimensional inviscid flow between two parallel walls. Generalizations have extended the theory to the quasi-geostrophic beta-plane (Charney and Stern 1962), and the semi-geostrophic β-plane (Eliassen 1983). This chapter further generalizes the theory to the quasi-geostrophic sphere. Unfortunately these advancements have only managed to produce generalized necessary conditions for instability. Without knowing a sufficient condition for instability in a flow, the application of these theories will be limited.

To derive the necessary condition for instability the governing equations are first linearized. Next a suitable solution must be assumed and substituted into the governing equations. The result is integrated over the domain, and the remaining integral relationships, combined with the appropriate boundary conditions, imply the necessary conditions for instability of the flow. Previous work developing the theory of barotropic-baroclinic instability has utilized two predominant assumptions for the form of the solution to the governing equations. An exponentially time dependent solution was assumed by Charney and Stern, while Taylor and Eliassen assumed the perturbations could be represented by a meridional displacement. This second method is slightly more general and also makes the dependence of the instability on the conservation of momentum explicit. In the following two sections, both of these methods will be generalized to a global quasi-balanced framework. We begin by linearizing the governing equations.

Recall the invertibility principle and the corresponding boundary conditions from chapters 2 and 3:

$$q = \nabla^2\psi + 4\Omega^2\mu^2 \frac{\partial}{\rho\partial z}\left(\frac{\rho}{N^2}\frac{\partial\psi}{\partial z}\right). \tag{4.1}$$

$$\frac{\partial\psi}{\partial z} = 0 \quad \text{at } z = z_T, \qquad 4\Omega^2\mu^2\left(\frac{\partial\psi}{\partial z} - \frac{N^2}{g}\psi\right) = B \quad \text{at } z = 0. \tag{4.2}$$

Also from chapter 2, the pseudo potential vorticity equation (2.24):

$$\frac{\partial q}{\partial t} + \frac{1}{a^2}\frac{\partial(\psi, q)}{\partial(\lambda, \mu)} + \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} = 0, \tag{4.3}$$

We now seek to linearize these equations by assuming the total fields are composed of a basic zonal state dependent on latitude and height plus a perturbation. For example

$$\psi(\lambda, \mu, z, t) = \bar{\psi}(\mu, z) + \psi'(\lambda, \mu, z, t), \tag{4.4}$$

and

$$q(\lambda, \mu, z, t) = \bar{q}(\mu, z) + q'(\lambda, \mu, z, t). \tag{4.5}$$

After plugging this form of ψ into (4.1) and neglecting terms containing the products of perturbations the pseudo-potential vorticity can thus be written as

$$q = -\frac{\partial(\bar{u}\cos\phi)}{a\cos\phi\partial\phi} + \nabla^2\psi' + 4\Omega^2\mu^2\frac{\partial}{\rho\partial z}\left(\frac{\rho}{N^2}\frac{\partial(\bar{\psi}+\psi')}{\partial z}\right). \tag{4.6}$$

Thus

$$\bar{q} = -\frac{\partial(\bar{u}\cos\phi)}{a\cos\phi\partial\phi} + 4\Omega^2\mu^2\frac{\partial}{\rho\partial z}\left(\frac{\rho}{N^2}\frac{\partial\bar{\psi}}{\partial z}\right), \tag{4.7}$$

and

$$q' = \nabla^2\psi' + 4\Omega^2\mu^2\frac{\partial}{\rho\partial z}\left(\frac{\rho}{N^2}\frac{\partial\psi'}{\partial z}\right). \tag{4.8}$$

After plugging these expressions into (4.3) and (2.25) we obtain the linearized governing equations including the boundary conditions, for the global quasi-balanced theory:

$$\left(\frac{\partial}{\partial t} + \bar{\omega}\frac{\partial}{\partial\lambda}\right)q' + v'\frac{\partial\bar{q}}{a\partial\phi} + \frac{2\Omega}{a^2}\frac{\partial\psi'}{\partial\lambda} = 0, \tag{4.9}$$

$$\frac{\partial B'}{\partial t} + \frac{1}{a^2}\left(\frac{\partial\psi'}{\partial\lambda}\frac{\partial\bar{B}}{\partial\mu} - \frac{\partial B'}{\partial\lambda}\frac{\partial\bar{\psi}}{\partial\mu}\right) = 0. \tag{4.10}$$

where the basic state zonal velocity $\bar{u}(\phi, z)$ is related to the basic state angular velocity $\bar{\omega}(\phi, z)$ by $\bar{u} = a\cos\phi\,\bar{\omega}$. The linearized boundary conditions are given by

$$\frac{\partial\psi'}{\partial z} = 0 \quad \text{at } z = z_T, \qquad 4\Omega^2\mu^2\left(\frac{\partial\psi'}{\partial z} - \frac{N^2}{g}\psi'\right) = B' \quad \text{at } z = 0. \tag{4.11}$$

Equation (4.9) is our final linearized equation that expresses the conservation of the pseudo-potential vorticity.

4.2 Derivation Assuming a Northward Parcel Displacement

After obtaining a linearized form of the pseudo-potential vorticity equation Taylor (1915) and Eliassen (1983) defined a parcel displacement η in the 'cross-stream' direction, which for us is the meridional (ϕ) direction. This allows us to forgo assuming a solution which depends exponentially on time. This method of deriving the necessary condition for instability also provides additional insight into the physical mechanisms that drive the instability.

Thus we can write

$$v' = \frac{\mathcal{D}\eta}{\mathcal{D}t}, \tag{4.12}$$

where

$$\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\partial}{\partial t} + \bar{\omega}\frac{\partial}{\partial \lambda}. \tag{4.13}$$

Equation (4.12) expresses the material movement of a fluid parcel in the meridional direction. However, the derivative operator in (4.12) is not strictly a material operator. The operator includes the time rate of change and the changes to the parcel in the longitudinal direction, but it does not include the meridional changes. Meridional displacements are contained in η. Substituting equation (4.12) into (4.9) results in

$$\frac{\mathcal{D}}{\mathcal{D}t}\left\{q' + \eta\left(\frac{2\Omega\cos\phi}{a} + \frac{\partial\bar{q}}{a\partial\phi}\right)\right\} = 0. \tag{4.14}$$

After integrating we are left with

$$q' = -\left(\frac{2\Omega\cos\phi}{a} + \frac{\partial\bar{q}}{a\partial\phi}\right)\eta. \tag{4.15}$$

We now examine the following equation:

$$\frac{\mathcal{D}}{\mathcal{D}t}\left(-\left(\frac{2\Omega\cos\phi}{a} + \frac{\partial\bar{q}}{a\partial\phi}\right)\frac{1}{2}\eta^2\right) = -\left(\frac{2\Omega\cos\phi}{a} + \frac{\partial\bar{q}}{a\partial\phi}\right)\eta\frac{\mathcal{D}\eta}{\mathcal{D}t} = v'q'. \tag{4.16}$$

Using the previously defined expressions for u' and v' in (4.8) we can rewrite q' as

$$q' = \frac{\partial v'}{a\cos\phi\partial\lambda} - \frac{\partial(u'\cos\phi)}{a\cos\phi\partial\phi} + 4\Omega^2\mu^2\frac{\partial}{\rho\partial z}\left(\frac{\rho}{N^2}\frac{\partial\psi'}{\partial z}\right), \tag{4.17}$$

so that

$$v'q' = v'\frac{\partial v'}{a\cos\phi\partial\lambda} - v'\cos\phi\frac{\partial(u'\cos\phi)}{a\cos^2\phi\partial\phi} + 4v'\Omega^2\mu^2\frac{\partial}{\rho\partial z}\left(\frac{\rho}{N^2}\frac{\partial\psi'}{\partial z}\right), \tag{4.18}$$

which can be rewritten as

$$v'q' = \frac{\partial}{a\cos\phi\partial\lambda}\left(\frac{1}{2}v'^2\right) - \frac{\partial(v'u'\cos^2\phi)}{a\cos^2\phi\partial\phi} + u'\cos\phi\frac{\partial(v'\cos\phi)}{a\cos^2\phi\partial\phi} + 4\Omega^2\mu^2v'\frac{\partial}{\rho\partial z}\left(\frac{\rho}{N^2}\frac{\partial\psi'}{\partial z}\right) \tag{4.19}$$

Utilizing the fact that (u', v') is nondivergent leads to

$$v'q' = \frac{\partial}{a\cos\phi\partial\lambda}\left(\frac{1}{2}(v'^2 - u'^2)\right) - \frac{\partial(v'u'\cos^2\phi)}{a\cos^2\phi\partial\phi} + 4v'\Omega^2\mu^2\frac{\partial}{\rho\partial z}\left(\frac{\rho}{N^2}\frac{\partial\psi'}{\partial z}\right). \tag{4.20}$$

The product rule can be used to rewrite the last term:

$$v'q' = \frac{\partial}{a\cos\phi\partial\lambda}\left(\frac{1}{2}(v'^2 - u'^2) - \frac{2\Omega^2\mu^2}{N^2}\left(\frac{\partial\psi'}{\partial z}\right)^2\right) - \frac{\partial(v'u'\cos^2\phi)}{a\cos^2\phi\partial\phi} + \frac{\partial}{\rho\partial z}\left(\frac{4\Omega^2\mu^2}{N^2}\rho v'\frac{\partial\psi'}{\partial z}\right). \tag{4.21}$$

Because our domain is spherical, continuous flow must be periodic in λ. The longitudinal average is defined as follows and will be denoted by square brackets:

$$[x] = \frac{1}{2\pi}\int_0^{2\pi} x\,d\lambda. \tag{4.22}$$

Plugging the expression for $v'q'$ into (4.16) and taking the average we obtain

$$-\left(\frac{2\Omega\cos\phi}{a} + \frac{\partial\bar{q}}{a\partial\phi}\right)\frac{1}{2}\frac{\partial[\eta^2]}{\partial t} = -\frac{\partial([v'u']\cos^2\phi)}{a\cos^2\phi\,\partial\phi} + \frac{\partial}{\rho\partial z}\left(\frac{4\Omega^2\mu^2}{N^2}\rho\left[v'\frac{\partial\psi'}{\partial z}\right]\right). \tag{4.23}$$

The first term on the right side of the previous equation contains the term $[v'u']\cos^2\phi$, which vanishes at the poles. Anticipating this simplification, we take the mass integral of (4.23) over height and from the south to the north pole:

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_0^{z_t}\int_{-\pi/2}^{\pi/2}&\left(\frac{2\Omega\cos\phi}{a} + \frac{\partial\bar{q}}{a\partial\phi}\right)[\eta^2]a\rho\cos^2\phi\,d\phi dz \\ &- \int_{-\pi/2}^{\pi/2}\left(\frac{4\Omega^2\mu^2}{N^2}\rho\left[v'\frac{\partial\psi'}{\partial z}\right]\right)_{z=0} a\cos^2\phi\,d\phi = 0.\end{aligned} \tag{4.24}$$

This is the generalized Charney-Stern necessary condition for combined barotropic-baroclinic instability on the sphere. Typically, on the f-plane, Eady type instability (pure boundary type instability) can occur when the 'northward' derivative of potential vorticity vanishes (in our case, when $2\Omega\cos\phi/a + \partial\bar{q}/a\partial\phi = 0$). It is unlikely this will equal zero under normal atmospheric conditions. Thus, it appears that Eady type instability is rarely, if ever applicable to a global quasi-balanced model.

If the variable I is defined to be

$$I = \frac{1}{2}\int_0^{z_t}\int_{-\pi/2}^{\pi/2}\left(2\Omega\cos\phi + \frac{\partial\bar{q}}{\partial\phi}\right)[\eta^2]\cos^2\phi\,d\phi dz \tag{4.25}$$

we can then write (4.24) as

$$\frac{dI}{dt} = \int_{-\pi/2}^{\pi/2} \left(\frac{4\Omega^2\mu^2}{N^2} \rho \left[v' \frac{\partial \psi'}{\partial z} \right] \right)_{z=0} a \cos^2 \phi \, d\phi. \tag{4.26}$$

For the case where the possible instability is completely internal we have:

$$\frac{dI}{dt} = 0. \tag{4.27}$$

Thus, for $\left[\eta^2\right]$ to grow in time, $2\Omega \cos\phi + \partial\bar{q}/\partial\phi$ must have both signs in the domain.

4.3 Derivation Assuming an Exponential Solution

Utilizing (4.9)-(4.10), we can also derive the Charney-Stern necessary condition for instability assuming an exponential solution of the form

$$\psi'(\lambda, \phi, z, t) = \Psi(\phi, z) e^{i(m\lambda - \nu t)}. \tag{4.28}$$

After plugging this solution into (4.9) and recalling that $\mu = \sin\phi$, and thus $d\mu = \cos\phi \, d\phi$, we arrive at

$$\left(\bar{\omega} - \frac{\nu}{m}\right) \left[\frac{\partial}{\partial \mu} \left((1-\mu^2) \frac{\partial \Psi}{\partial \mu} \right) - \frac{m^2 \Psi}{(1-\mu^2)} + 4\Omega^2 a^2 \mu^2 \frac{\partial}{\rho \partial z} \left(\frac{\rho}{N^2} \frac{\partial \Psi}{\partial z} \right) \right] + \Psi \left(2\Omega + \frac{\partial \bar{q}}{\partial \mu} \right) = 0, \tag{4.29}$$

which can be written in a more instructive form as

$$\frac{\partial}{\partial \mu} \left((1-\mu^2) \frac{\partial \Psi}{\partial \mu} \right) - \frac{m^2 \Psi}{(1-\mu^2)} + 4\Omega^2 a^2 \mu^2 \frac{\partial}{\rho \partial z} \left(\frac{\rho}{N^2} \frac{\partial \Psi}{\partial z} \right) + \left(2\Omega + \frac{\partial \bar{q}}{\partial \mu} \right) \frac{(\bar{\omega} - \nu^*/m)}{|\bar{\omega} - \nu/m|^2} \Psi = 0. \tag{4.30}$$

The condition for instability we are deriving is ultimately an integral constraint that describes the distribution of the meridional gradient of the pseudo potential vorticity; thus we will eventually need to perform an integration on (4.30). We expect this to provide the desired information. The following operations involving the complex conjugate of Ψ and taking the difference between the resulting two equations are a mathematical trick that has been developed that conveniently eliminates a few terms which would be troublesome to integrate.

The complex conjugate of (4.30) is

$$\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Psi^*}{\partial\mu}\right)-\frac{m^2\Psi^*}{(1-\mu^2)}+4\Omega^2a^2\mu^2\frac{\partial}{\rho\partial z}\left(\frac{\rho}{N^2}\frac{\partial\Psi^*}{\partial z}\right)+\left(2\Omega+\frac{\partial\bar{q}}{\partial\mu}\right)\frac{(\bar{\omega}-\nu/m)}{|(\bar{\omega}-\nu/m)|^2}\Psi^*=0. \tag{4.31}$$

Multiplying (4.30) by Ψ^* and (4.31) by Ψ and then taking the difference of these two equations yields

$$\frac{\partial}{\partial\mu}\left[(1-\mu^2)\left(\Psi^*\frac{\partial\Psi}{\partial\mu}-\Psi\frac{\partial\Psi^*}{\partial\mu}\right)\right]+4\Omega^2a^2\mu^2\frac{\partial}{\rho\partial z}\left[\frac{\rho}{N^2}\left(\Psi^*\frac{\partial\Psi}{\partial z}-\Psi\frac{\partial\Psi^*}{\partial z}\right)\right] + \left(2\Omega+\frac{\partial\bar{q}}{\partial\mu}\right)\frac{2i\nu_i}{|(\bar{\omega}-\nu/m)|^2}|\Psi|^2=0. \tag{4.32}$$

This equation is now in a form which can be usefully integrated. Taking the mass integral of (4.32) gives a somewhat simplified expression:

$$-\int_{-1}^{1}4\Omega^2a^2\mu^2\left[\frac{\rho}{N^2}\left(\Psi^*\frac{\partial\Psi}{\partial z}-\Psi\frac{\partial\Psi^*}{\partial z}\right)\right]d\mu + \int_{z_0}^{z_t}\int_{-1}^{1}\left(2\Omega+\frac{\partial\bar{q}}{\partial\mu}\right)\frac{2i\nu_i}{|(\bar{\omega}-\nu/m)|^2}|\Psi|^2\rho d\mu dz=0, \tag{4.33}$$

where $\partial\Psi/\partial z=0$ and $\partial\Psi^*/\partial z=0$ at $z=z_t$ and the first term on the left has been evaluated at $z=0$. It can be shown that at $z=0$

$$\Psi^*\frac{\partial\Psi}{\partial z}=-\frac{\partial\bar{B}}{\partial\mu}\frac{(\bar{\omega}-\nu^*/m)}{|\bar{\omega}-\nu/m|^2}\frac{|\Psi|^2}{4\Omega^2a^2\mu^2}+\frac{N^2|\Psi|^2}{g}. \tag{4.34}$$

and

$$\Psi\frac{\partial\Psi^*}{\partial z}=-\frac{\partial\bar{B}}{\partial\mu}\frac{(\bar{\omega}-\nu/m)}{|\bar{\omega}-\nu/m|^2}\frac{|\Psi|^2}{4\Omega^2a^2\mu^2}+\frac{N^2|\Psi|^2}{g}, \tag{4.35}$$

If we now substitute these equations into the first term in (4.33) we obtain

$$-\int_{-1}^{1}4\Omega^2a^2\mu^2\left[\frac{\rho}{N^2}\left(\frac{\partial\bar{B}}{\partial\mu}\frac{\nu^*/m}{|\bar{\omega}-\nu/m|^2}\frac{|\Psi|^2}{4\Omega^2a^2\mu^2}-\frac{\partial\bar{B}}{\partial\mu}\frac{\nu/m}{|\bar{\omega}-\nu/m|^2}\frac{|\Psi|^2}{4\Omega^2a^2\mu^2}\right)\right]d\mu + \int_{0}^{z_t}\int_{-1}^{1}\left(2\Omega+\frac{\partial\bar{q}}{\partial\mu}\right)\frac{2i\nu_i}{|\bar{\omega}-\nu/m|^2}|\Psi|^2\rho\,d\mu\,dz=0. \tag{4.36}$$

Upon simplification, (4.36) becomes

$$\int_{-1}^{1}\frac{\rho}{N^2}\frac{\partial\bar{B}}{\partial\mu}\frac{2i\nu_i}{|\bar{\omega}-\nu/m|^2}|\Psi|^2\,d\mu+\int_{0}^{z_t}\int_{-1}^{1}\left(2\Omega+\frac{\partial\bar{q}}{\partial\mu}\right)\frac{2i\nu_i}{|\bar{\omega}-\nu/m|^2}|\Psi|^2\rho\,d\mu\,dz=0, \tag{4.37}$$

which can also be written as

$$\nu_i \left\{ \int_{-1}^{1} \frac{\rho}{N^2} \frac{\partial \bar{B}}{\partial \mu} \frac{|\Psi|^2}{|\bar{\omega} - \nu/m|^2} \, d\mu + \int_0^{z_t} \int_{-1}^{1} \left(2\Omega + \frac{\partial \bar{q}}{\partial \mu} \right) \frac{|\Psi|^2}{|\bar{\omega} - \nu/m|^2} \rho \, d\mu \, dz \right\} = 0. \qquad (4.38)$$

Recall we are trying to generalize the criterion for which a flow will be unstable to small perturbations. Due to the particular form of the solution (4.28) we are using, the growth of a wave is represented by its imaginary frequency ν_i. For unstable flows we require ν_i in the above equation to be non-zero. This implies the term inside the braces must be zero. Because ρ/N^2 and $|\Psi|^2/|\bar{\omega} - \nu/m|^2$ are obviously positive we can infer from (4.38) that $\partial\bar{B}/\partial\mu$ and $\partial\bar{q}/\partial\mu$ must change signs for the flow to be unstable. This is the generalized condition for combined baroclinic-barotropic instability on a sphere.

When compared with the analogous expression for Charney and Stern's instability condition on the beta plane it can be seen that the spherical generalization above matches very well. The primary significant difference concerns the so-called Eady-type instability as discussed in the previous section.

4.4 The Two-Layer, Quasi-Balanced Model on the Sphere

4.4.1 Derivation

Consider a fluid system containing two layers of homogeneous incompressible fluid, with the lower layer having density ρ_1 and the upper layer having density $\rho_2 = \gamma\rho_1$, with the constant $\gamma < 1$. The global mean depths of the two layers are assumed to be equal and are denoted by the constant $\bar{h}$. The deviations from the mean depth in the respective layers ($j = 1, 2$) are denoted by $h_j(\lambda, \mu, t)$, where λ is the longitude and ϕ the latitude. In the context of the inviscid primitive equations on the sphere, the potential vorticity equations for the two layers are $D_j P_j/Dt = 0$, where $D_j/Dt = \partial/\partial t + u_j(\partial/a\cos\phi\partial\lambda) + v_j(\partial/a\partial\phi)$ is the material derivative, u_j and v_j the zonal and meridional velocity components, a the radius of the Earth, $P_j = (2\Omega\sin\phi + \nabla^2\psi_j)/(\bar{h} + h_j)$ the potential vorticity, Ω the angular rotation rate of the Earth, ∇^2 the horizontal Laplacian operator, and ψ_j the streamfunction for the rotational part of the flow. As a first step in the derivation of the two layer quasi-balanced model, we approximate P_j by the linear form

$$P_j \approx \frac{1}{\bar{h}}\left(2\Omega\sin\phi + \nabla^2\psi_j - \frac{2\Omega\sin\phi}{\bar{h}}h_j\right). \tag{4.39}$$

Equation (4.39) is not yet in the form of an invertibility principle since both ψ_j and h_j appear on the right hand side. It can be converted into an invertibility principle by introducing an approximate balance relation between the wind and mass fields, i.e., a relation between ψ_j and h_j.

To find a relation between ψ_j and h_j, we use a similar argument as was used in Chapter 3. At an arbitrary height z, the pressure is given hydrostatically by $p_1 = \rho_1 g(\bar{h}+h_1-z)+\rho_2 g(\bar{h}+h_2)$ for the lower layer and $p_2 = \rho_2 g(\bar{h}+h_1+\bar{h}+h_2-z)$ for the upper layer, so that the pressure gradient forces in the respective layers are $\rho_1^{-1}\nabla p_1 = g\nabla(h_1+\gamma h_2)$ and $\rho_2^{-1}\nabla p_2 = g\nabla(h_1+h_2)$. The linear balance relations for the two layers are $\nabla\cdot(2\Omega\mu\nabla\psi_1) = g\nabla^2(h_1+\gamma h_2)$ and $\nabla\cdot(2\Omega\mu\nabla\psi_2) = g\nabla^2(h_1+h_2)$, where $\mu = \sin\phi$. Considering $2\Omega\mu$ to be slowly varying, we can approximate the linear balance relations by $\nabla^2[g(h_1+\gamma h_2)-2\Omega\mu\psi_1] = 0$ and $\nabla^2[g(h_1+h_2)-2\Omega\mu\psi_2] = 0$, from which it follows that $g(h_1+\gamma h_2) = 2\Omega\mu\psi_1$ and $g(h_1+h_2) = 2\Omega\mu\psi_2$, or

$$h_1 = \frac{2\Omega\mu}{g(1-\gamma)}\left(\psi_1 - \gamma\psi_2\right), \tag{4.40}$$

$$h_2 = \frac{2\Omega\mu}{g(1-\gamma)}\left(\psi_2 - \psi_1\right). \tag{4.41}$$

Using these in (4.39) and defining $q_j = \bar{h}P_j - 2\Omega\mu$, we obtain the invertibility relations given below in (4.44) and (4.45). Approximating the advecting velocity components in the material derivative operator by the rotational components $-\partial\psi_j/a\partial\phi$ and $\partial\psi_j/a\cos\phi\partial\lambda$, the potential vorticity principles become (4.42) and (4.43) below. To summarize, the two-layer quasi-balanced model on the sphere is

$$\frac{\partial q_1}{\partial t} - \frac{\partial\psi_1}{a\partial\mu}\frac{\partial q_1}{a\partial\lambda} + \frac{\partial\psi_1}{a\partial\lambda}\left(\frac{2\Omega}{a} + \frac{\partial q_1}{a\partial\mu}\right) = 0, \tag{4.42}$$

$$\frac{\partial q_2}{\partial t} - \frac{\partial\psi_2}{a\partial\mu}\frac{\partial q_2}{a\partial\lambda} + \frac{\partial\psi_2}{a\partial\lambda}\left(\frac{2\Omega}{a} + \frac{\partial q_2}{a\partial\mu}\right) = 0, \tag{4.43}$$

$$q_1 = \nabla^2\psi_1 - \frac{4\Omega^2\mu^2}{g(1-\gamma)\bar{h}}\left(\psi_1 - \gamma\psi_2\right) \tag{4.44}$$

$$q_2 = \nabla^2\psi_2 - \frac{4\Omega^2\mu^2}{g(1-\gamma)\bar{h}}\left(\psi_2 - \psi_1\right). \tag{4.45}$$

Equations (4.42)–(4.45) constitute a system of four equations in the four unknowns $q_1(\lambda, \mu, t)$, $q_2(\lambda, \mu, t)$, $\psi_1(\lambda, \mu, t)$ and $\psi_2(\lambda, \mu, t)$. Note that (4.44) and (4.45) are coupled in such a way that a potential vorticity anomaly in one layer induces a flow in both layers. For the case of Rossby-Haurwitz waves, we shall solve (4.44) and (4.45) using spheroidal harmonic transforms. However, before this can be accomplished, (4.44) and (4.45) must be combined in certain ways that lead to two decoupled equations. This is accomplished using a vertical normal mode transform, which is discussed in the next section.

4.4.2 Vertical Normal Mode Transform

Introducing the vector notation $\mathbf{q} = [q_1, q_2]^T$ and $\boldsymbol{\psi} = [\psi_1, \psi_2]^T$, we can write (4.44) and (4.45) in the vector form

$$\nabla^2 \boldsymbol{\psi} - \frac{4\Omega^2 \mu^2}{g} \mathbf{E} \boldsymbol{\psi} = \mathbf{q}, \tag{4.46}$$

where

$$\mathbf{E} = \frac{1}{(1-\gamma)\bar{h}} \begin{pmatrix} 1 & -\gamma \\ -1 & 1 \end{pmatrix}. \tag{4.47}$$

We now transform the invertibility principle from its layer-space representation (4.46) to its vertical mode representation. The vertical mode representations for the potential vorticity and streamfunction are $\mathbf{Q} = [Q^{(0)}, Q^{(1)}]^T$ and $\mathbf{\Psi} = [\Psi^{(0)}, \Psi^{(1)}]^T$, where the superscript "0" denotes the barotropic mode and the superscript "1" denotes the first baroclinic mode. The transform pair for the potential vorticity is

$$\mathbf{Q} = \mathbf{V}\mathbf{M}^{1/2}\mathbf{q}, \qquad \mathbf{q} = \mathbf{M}^{-1/2}\mathbf{V}^{-1}\mathbf{Q}, \tag{4.48}$$

and for the streamfunction is

$$\mathbf{\Psi} = \mathbf{V}\mathbf{M}^{1/2}\boldsymbol{\psi}, \qquad \boldsymbol{\psi} = \mathbf{M}^{-1/2}\mathbf{V}^{-1}\mathbf{\Psi}, \tag{4.49}$$

where the "mass matrix" $\mathbf{M}$ is defined by

$$\mathbf{M} = \frac{1}{1+\gamma} \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}, \tag{4.50}$$

and the "vertical transform" matrix $\mathbf{V}$ is as yet unspecified.

To transform the invertibility principle (4.46) we first multiply it by $\mathbf{V}\mathbf{M}^{1/2}$, and then use (4.48) and (4.49) to obtain

$$\nabla^2\mathbf{\Psi} - \frac{4\Omega^2\mu^2}{g}\mathbf{V}\mathbf{A}\mathbf{V}^{-1}\mathbf{\Psi} = \mathbf{Q}, \tag{4.51}$$

where $\mathbf{A} = \mathbf{M}^{1/2}\mathbf{E}\mathbf{M}^{-1/2}$. Using (4.47) and (4.50) we can easily perform the matrix multiplication $\mathbf{M}^{1/2}\mathbf{E}\mathbf{M}^{-1/2}$ to obtain

$$\mathbf{A} = \frac{1}{(1-\gamma)\bar{h}}\begin{pmatrix} 1 & -\gamma^{1/2} \\ -\gamma^{1/2} & 1 \end{pmatrix}. \tag{4.52}$$

We now construct $\mathbf{V}$ out of the eigenvectors of $\mathbf{A}$. Since $\mathbf{A}$ is symmetric, its eigenvalues are real and its eigenvectors are orthogonal. The determinant of $\mathbf{A}$ is positive provided $\gamma < 1$. Thus, as long as the fluid is stably stratified, $\mathbf{A}$ is positive definite. This guarantees that the eigenvalues of $\mathbf{A}$ are positive. Let $\mathbf{x}^{(k)}$ be the normalized eigenvector of $\mathbf{A}$ with associated eigenvalue $1/h^{(k)}$, i.e.,

$$\mathbf{A}\mathbf{x}^{(k)} = \frac{1}{h^{(k)}}\mathbf{x}^{(k)} \qquad \text{for} \quad k = 0, 1 \tag{4.53}$$

with

$$\left[\mathbf{x}^{(k)}\right]^T \mathbf{x}^{(\ell)} = \begin{cases} 1 & \text{if} \quad k = \ell \\ 0 & \text{if} \quad k \neq \ell. \end{cases} \tag{4.54}$$

The solutions of (4.53) satisfying the normalization condition (4.54) are

$$\mathbf{x}^{(0)} = 2^{-1/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{for} \qquad h^{(0)} = \left(1+\gamma^{1/2}\right)\bar{h}, \tag{4.55}$$

and

$$\mathbf{x}^{(1)} = 2^{-1/2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{for} \qquad h^{(1)} = \left(1-\gamma^{1/2}\right)\bar{h}. \tag{4.56}$$

Let the two rows of $\mathbf{V}$ consist of the two eigenvectors $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$. Because of the orthonormality relation (4.54), $\mathbf{V}^T\mathbf{V} = \mathbf{I}$, so that $\mathbf{V}$ is an orthogonal matrix and $\mathbf{V}^{-1} = \mathbf{V}^T$. Thus,

$$\mathbf{V} = \mathbf{V}^T = \mathbf{V}^{-1} = 2^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4.57}$$

From (4.52) and (4.57) we obtain

$$\mathbf{VAV}^{-1} = \mathbf{VAV}^{T} = \begin{pmatrix} 1/\bar{h}^{(0)} & 0 \\ 0 & 1/\bar{h}^{(1)} \end{pmatrix}, \tag{4.58}$$

which simplifies the vector equation (4.51) into the equivalent scalar forms given below in (4.60) and (4.61). In addition, the first entry in (4.48) can be written in the scalar form (4.59) and the last entry in (4.49) can be written in the scalar form (4.62).

To summarize, the invertibility principle now takes the form

$$\begin{pmatrix} Q^{(0)} \\ Q^{(1)} \end{pmatrix} = \frac{1}{2^{1/2}(1+\gamma)^{1/2}} \begin{pmatrix} 1 & \gamma^{1/2} \\ 1 & -\gamma^{1/2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{4.59}$$

$$\nabla^2 \Psi^{(0)} - \frac{\epsilon^{(0)}\mu^2}{a^2}\Psi^{(0)} = Q^{(0)}, \tag{4.60}$$

$$\nabla^2 \Psi^{(1)} - \frac{\epsilon^{(1)}\mu^2}{a^2}\Psi^{(1)} = Q^{(1)}, \tag{4.61}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{(1+\gamma)^{1/2}}{2^{1/2}} \begin{pmatrix} 1 & 1 \\ \gamma^{-1/2} & -\gamma^{-1/2} \end{pmatrix} \begin{pmatrix} \Psi^{(0)} \\ \Psi^{(1)} \end{pmatrix}, \tag{4.62}$$

where the constants $\epsilon^{(0)} = 4\Omega^2 a^2/(gh^{(0)})$ and $\epsilon^{(1)} = 4\Omega^2 a^2/(gh^{(1)})$ are the Lamb's parameters for the external and first internal modes. Knowing the layer values $q_1(\lambda,\mu)$ and $q_2(\lambda,\mu)$ from the prediction equations for PV, we first compute the vertical modes $Q^{(0)}(\lambda,\mu)$ and $Q^{(1)}(\lambda,\mu)$ using (4.59). We then solve the decoupled invertibility principle (4.60) and (4.61) for the streamfunction vertical modes $\Psi^{(0)}(\lambda,\mu)$ and $\Psi^{(1)}(\lambda,\mu)$. Finally, we transform $\Psi^{(0)}(\lambda,\mu)$ and $\Psi^{(1)}(\lambda,\mu)$ to the layer representation $\psi_1(\lambda,\mu)$ and $\psi_2(\lambda,\mu)$ using (4.62). Equations (4.60) and (4.61) are amenable to solution via spheroidal harmonic expansions because the spheroidal harmonic $S_{mn}(\lambda,\mu;\epsilon)$ is an eigenfunction of the operator $\nabla^2 - \epsilon\mu^2/a^2$.

Concerning typical values of the Lamb's parameters, when $\bar{h} = 5000\,\text{m}$ and $\gamma = 0.9$, we obtain $h^{(0)} = \left(1+\gamma^{1/2}\right)\bar{h} = 9743\,\text{m}$ and $h^{(1)} = \left(1-\gamma^{1/2}\right)\bar{h} = 256.6\,\text{m}$, so that $gh^{(0)} = (309.0\,\text{ms}^{-1})^2$ and $gh^{(1)} = (50.14\,\text{ms}^{-1})^2$, which results in $\epsilon^{(0)} = 4\Omega^2 a^2/(gh^{(0)}) = 9.04$ and $\epsilon^{(1)} = 4\Omega^2 a^2/(gh^{(1)}) = 343$.

4.4.3 Rossby-Haurwitz Wave Dispersion

We now linearize (4.42) and (4.43) about a resting basic state. Thus, $q_1(\lambda,\mu,t) = q_1'(\lambda,\mu,t)$ and $\psi_1(\lambda,\mu,t) = \psi_1'(\lambda,\mu,t)$, with similar relations for $q_2(\lambda,\mu,t)$ and $\psi_2(\lambda,\mu,t)$. The linearized versions of (4.42) and (4.43) are

$$\frac{\partial q_1'}{\partial t} + \frac{2\Omega}{a^2}\frac{\partial \psi_1'}{\partial \lambda} = 0, \tag{4.63}$$

$$\frac{\partial q_2'}{\partial t} + \frac{2\Omega}{a^2}\frac{\partial \psi_2'}{\partial \lambda} = 0. \tag{4.64}$$

Equations (4.63) and (4.64) can be written in the vector form

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{2\Omega}{a^2}\frac{\partial \boldsymbol{\psi}}{\partial \lambda} = 0. \tag{4.65}$$

Multiplying (4.65) by $\mathbf{VM}^{1/2}$, we obtain

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{2\Omega}{a^2}\frac{\partial \boldsymbol{\Psi}}{\partial \lambda} = 0, \tag{4.66}$$

which, with the invertibility principle, can be written as

$$\frac{\partial}{\partial t}\left(\nabla^2\Psi^{(0)} - \frac{\epsilon^{(0)}\mu^2}{a^2}\Psi^{(0)}\right) + \frac{2\Omega}{a^2}\frac{\partial \Psi^{(0)}}{\partial \lambda} = 0, \tag{4.67}$$

$$\frac{\partial}{\partial t}\left(\nabla^2\Psi^{(1)} - \frac{\epsilon^{(1)}\mu^2}{a^2}\Psi^{(1)}\right) + \frac{2\Omega}{a^2}\frac{\partial \Psi^{(1)}}{\partial \lambda} = 0. \tag{4.68}$$

Note that (4.67) and (4.68) are a decoupled pair of equations in the unknowns $\Psi^{(0)}(\lambda,\mu,t)$ and $\Psi^{(1)}(\lambda,\mu,t)$.

We now search for solutions of (4.67) and (4.68) having the forms $\Psi^{(0)}(\lambda,\mu,t) = \hat{\Psi}^{(0)}e^{i(m\lambda-\nu t)}\mathcal{S}_{mn}(\mu;\epsilon^{(0)})$ and $\Psi^{(1)}(\lambda,\mu,t) = \hat{\Psi}^{(1)}e^{i(m\lambda-\nu t)}\mathcal{S}_{mn}(\mu;\epsilon^{(1)})$, where $\hat{\Psi}^{(0)}$ and $\hat{\Psi}^{(1)}$ are complex constants and $\mathcal{S}_{mn}(\mu;\epsilon^{(j)})$ are the spheroidal harmonics. Substituting these into (4.67) and (4.68) we obtain

$$\nu_{mn}^{(0)} = -\frac{2\Omega m}{\alpha_{mn}(\epsilon^{(0)})}, \qquad \nu_{mn}^{(1)} = -\frac{2\Omega m}{\alpha_{mn}(\epsilon^{(1)})}, \tag{4.69}$$

which are the Rossby-Haurwitz wave frequencies for the external and first internal modes.

4.4.4 Baroclinic Instability

In contrast to the previous section in which (4.42) and (4.43) were linearized about a resting basic state, (4.42) and (4.43) are now linearized about a zonally symmetric basic state. The streamfunction for this basic flow is assumed to be $\bar{\psi}_1(\mu) = a^2\bar{\omega}\mu$ and $\bar{\psi}_2(\mu) = -a^2\bar{\omega}\mu$, where $\bar{\omega}$ is a positive constant. The basic state angular velocity in the two layers is then given by $-(1/a^2)(d\bar{\psi}_1/d\mu) = -\bar{\omega}$ and $-(1/a^2)(d\bar{\psi}_2/d\mu) = \bar{\omega}$, i.e., a constant westerly angular velocity in the upper layer and a constant easterly angular velocity in the lower layer.

The linearized versions of (4.42)-(4.45) are

$$\frac{\partial q_1'}{\partial t} - \bar{\omega}\frac{\partial q_1'}{\partial \lambda} + \frac{1}{a^2}\frac{\partial \psi_1'}{\partial \lambda}\left(2\Omega + \frac{\partial \bar{q}_1}{\partial \mu}\right) = 0, \tag{4.70}$$

$$\frac{\partial q_2'}{\partial t} + \bar{\omega}\frac{\partial q_2'}{\partial \lambda} + \frac{1}{a^2}\frac{\partial \psi_2'}{\partial \lambda}\left(2\Omega + \frac{\partial \bar{q}_2}{\partial \mu}\right) = 0, \tag{4.71}$$

$$q_1' = \nabla^2\psi_1' - \frac{4\Omega^2\mu^2}{g(1-\gamma)\bar{h}}\left(\psi_1' - \gamma\psi_2'\right), \tag{4.72}$$

$$q_2' = \nabla^2\psi_2' - \frac{4\Omega^2\mu^2}{g(1-\gamma)\bar{h}}\left(\psi_2' - \psi_1'\right). \tag{4.73}$$

We now have four equations in the four unknowns $q_1'(\lambda,\mu,t)$, $q_2'(\lambda,\mu,t)$,$\psi_1'(\lambda,\mu,t)$, and $\psi_2'(\lambda,\mu,t)$.

When $q_1', \bar{q}_1, q_2'$ and $\bar{q}_2$ are substituted into (4.70) and (4.71) we can write (4.70) and (4.71) as

$$\left(\frac{\partial}{\partial t} - \bar{\omega}\frac{\partial}{\partial \lambda}\right)\left[\nabla^2\psi_1' - \frac{4\Omega^2\mu^2}{g(1-\gamma)\bar{h}}(\psi_1' - \gamma\psi_2')\right] + \frac{1}{a^2}\left[2\Omega - 2\bar{\omega}\left(1 + \frac{6(1+\gamma)\Omega^2 a^2\mu^2}{g(1-\gamma)\bar{h}}\right)\right]\frac{\partial \psi_1'}{\partial \lambda} = 0, \tag{4.74}$$

$$\left(\frac{\partial}{\partial t} + \bar{\omega}\frac{\partial}{\partial \lambda}\right)\left[\nabla^2\psi_2' - \frac{4\Omega^2\mu^2}{g(1-\gamma)\bar{h}}(\psi_2' - \psi_1')\right] + \frac{1}{a^2}\left[2\Omega + 2\bar{\omega}\left(1 + \frac{12\Omega^2 a^2\mu^2}{g(1-\gamma)\bar{h}}\right)\right]\frac{\partial \psi_2'}{\partial \lambda} = 0. \tag{4.75}$$

This now leaves us with two equations and two unknowns.

It can be shown that for the two-layer model on an f-plane, the reversal of the poleward gradient of basic state potential vorticity between the two layers allows for counter-propagating Rossby waves, which leads to baroclinic instability. This section attempts to derive the analogous result for the global case. The above two equations can be compared to (4.63-4.64). In the previous section, the next step was to convert the linearized equations into vector form. After writing the equations in vector form the vertical transform was then applied as a means of decoupling the two layers.

As can be seen in (4.74) and (4.75), the sign reversal that is necessary to generate baroclinic instability on the f-plane complicates the linearized equations of the global model to the extent that putting them into a suitable vector form proves difficult. Thus the linearized set of governing equations for the global case cannot be decoupled using the vertical normal mode method as was done for the Rossby-Haurwitz problem.

For the quasi-balanced two-layer model on an f-plane, the set of linearized equations analogous to (4.70)-(4.73) can be simplified to an algebraic system of four equations in four unknowns if solutions roughly of the form $q_1' = \hat{q}_1 e^{ik(x-ct)} \cos(\ell y)$, etc. are substituted into the linearized system. These algebraic equations, although still coupled, can be analytically solved to give a concise description of the interaction between the counter-propagating Rossby waves. Hoping for a similar simplification we now search for solutions of the form $\psi_1'(\lambda, \mu, t) = \hat{\psi}_1(\mu) e^{i(m\lambda - \nu t)}$ and $\psi_2'(\lambda, \mu, t) = \hat{\psi}_2(\mu) e^{i(m\lambda - \nu t)}$. Here the actual form of the functions $\hat{\psi}_1(\mu)$ and $\hat{\psi}_2(\mu)$ is undetermined. It was thought that (4.74) and (4.75) would be easily solvable through the use of spheroidal harmonics. Unfortunately, $\hat{\psi}_1(\mu)$ and $\hat{\psi}_2(\mu)$ did not turn out to match the form of spheroidal harmonics or any other recognizable standard functions. As a result, we have not managed to derive a simplified set of algebraic equations analogous to the f-plane case.

A Boussinesq-like approximation is now made by assuming γ to be 1 where it appears decoupled from gravity and leaving it as γ when it is coupled to gravity. This leads to the following equations

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{d\hat{\psi}_1}{d\mu}\right] - \frac{m^2}{1-\mu^2}\hat{\psi}_1 - \epsilon\mu^2\left(\hat{\psi}_1 - \hat{\psi}_2\right) = a^2\hat{q}_1, \tag{4.76}$$

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{d\hat{\psi}_2}{d\mu}\right] - \frac{m^2}{1-\mu^2}\hat{\psi}_2 + \epsilon\mu^2\left(\hat{\psi}_1 - \hat{\psi}_2\right) = a^2\hat{q}_2, \tag{4.77}$$

$$-\left(\frac{\nu}{m} + \bar{\omega}\right) a^2\hat{q}_1 + \left(2\Omega - 6\epsilon\mu^2\bar{\omega}\right)\hat{\psi}_1 = 0, \tag{4.78}$$

$$-\left(\frac{\nu}{m} - \bar{\omega}\right) a^2\hat{q}_2 + \left(2\Omega + 6\epsilon\mu^2\bar{\omega}\right)\hat{\psi}_2 = 0. \tag{4.79}$$

These equations can be cast in a more instructive light if they are arranged in terms of $\hat{\psi}_s = \hat{\psi}_1 + \hat{\psi}_2$ and $\hat{\psi}_d = \hat{\psi}_1 - \hat{\psi}_2$ where $\hat{\psi}_s$ can be thought of as a barotropic variable and $\hat{\psi}_d$ can be thought of as a baroclinic variable. Thus we obtain

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{d\hat{\psi}_s}{d\mu}\right] - \frac{m^2}{1-\mu^2}\hat{\psi}_s = a^2\hat{q}_s, \tag{4.80}$$

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{d\hat{\psi}_d}{d\mu}\right] - \frac{m^2}{1-\mu^2}\hat{\psi}_d - 2\epsilon\mu^2\hat{\psi}_d = a^2\hat{q}_d, \tag{4.81}$$

$$-a^2\left[\frac{\nu}{m}\hat{q}_s + \bar{\omega}\hat{q}_d\right] + 2\Omega\hat{\psi}_s - 6\epsilon\mu^2\bar{\omega}\hat{\psi}_d = 0, \tag{4.82}$$

$$-a^2\left[\frac{\nu}{m}\hat{q}_d + \bar{\omega}\hat{q}_s\right] + 2\Omega\hat{\psi}_d - 6\epsilon\mu^2\bar{\omega}\hat{\psi}_s = 0. \tag{4.83}$$

We now have four equations in the four unknowns $\hat{\psi}_s$, $\hat{\psi}_d$, $\hat{q}_s$, and $\hat{q}_d$. So far it is still not clear exactly how to solve these differential equations, however their organization strongly suggests a solution that combines both spherical and spheroidal harmonic functions. To further elucidate this we substitute $\hat{q}_s$ from (4.80) and $\hat{q}_d$ from (4.81) into (4.82) and (4.83). This then leaves us with two equations in the two unknowns $\hat{\psi}_s$ and $\hat{\psi}_d$

$$\begin{aligned}\frac{\nu}{m}&\left\{\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{d\hat{\psi}_s}{d\mu}\right] - \frac{m^2}{1-\mu^2}\hat{\psi}_s - \frac{2\Omega m}{\nu}\hat{\psi}_s\right\}\\ &+\bar{\omega}\left\{\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{d\hat{\psi}_d}{d\mu}\right] - \frac{m^2}{1-\mu^2}\hat{\psi}_d + 4\epsilon\mu^2\hat{\psi}_d\right\} = 0,\end{aligned} \tag{4.84}$$

$$\begin{aligned}\bar{\omega}&\left\{\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{d\hat{\psi}_s}{d\mu}\right] - \frac{m^2}{1-\mu^2}\hat{\psi}_s + 6\epsilon\mu^2\hat{\psi}_s\right\}\\ &+\frac{\nu}{m}\left\{\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{d\hat{\psi}_d}{d\mu}\right] - \frac{m^2}{1-\mu^2}\hat{\psi}_d - 2\epsilon\mu^2\hat{\psi}_d - \frac{2\Omega m}{\nu}\hat{\psi}_d\right\} = 0.\end{aligned} \tag{4.85}$$

It is interesting to note that the first term of (4.84) is the differential equation which leads to spherical harmonic solutions, while the second term in (4.84) and both terms in (4.85) are the differential equations leading to spheroidal harmonic solutions. Unfortunately, even with this apparent structure, it is still unclear how to proceed. Equations (4.84) and (4.85) represent an eigenvalue problem with four separate eigenfunctions.

$$\begin{pmatrix} 2\Omega & -\bar{\omega}(\nabla^2 + 4\epsilon\mu^2) \\ -\bar{\omega}(\nabla^2 + 6\epsilon\mu^2) & 2\Omega \end{pmatrix} \begin{pmatrix} \hat{\psi}_s \\ \hat{\psi}_d \end{pmatrix} = \frac{\nu}{m} \begin{pmatrix} \nabla^2 & 0 \\ 0 & \nabla^2 - 2\epsilon\mu^2 \end{pmatrix} \begin{pmatrix} \hat{\psi}_s \\ \hat{\psi}_d \end{pmatrix} \tag{4.86}$$

At this point it is believed that further analytical progress is not possible for the simplified, two-layer quasi-balanced global system. We can however solve these equations numerically through expanding them as series of associated Legendre functions. In spite of the lack of a concise analytic solution for this simplest of quasi-balanced global two-layer models, this work has not been fruitless. Rather, it demonstrates the complexity introduced by the curvature terms to the coupled two layer equations for this specific two-layer model. Although it has been made clear here that analytical solutions for a linearized, Boussinesq-like spherical two-layer model cannot be found; it is reasonable to assume that for a two-layer model with fewer approximations the solution could be obtained through the use of more complicated mathematical methods.

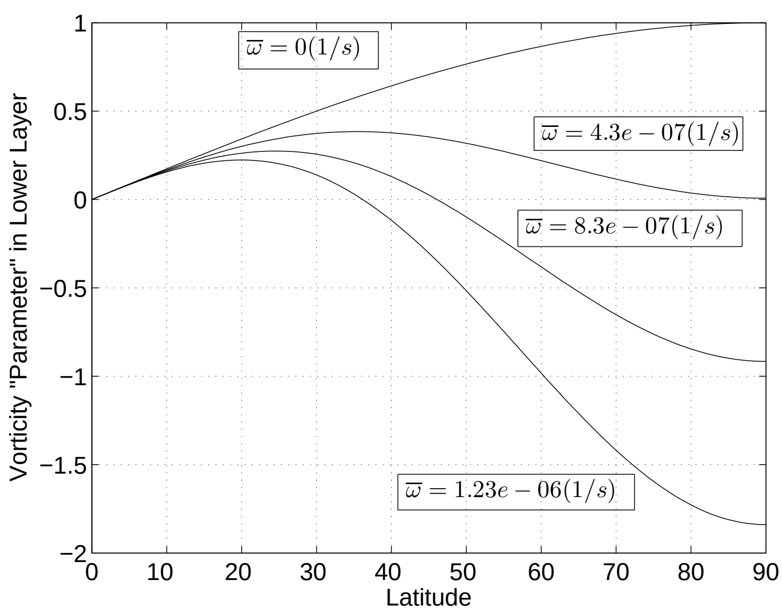

Figure 4.1: NonDimensionalized Family of Angular Frequency curves in the lower layer.

Even without a precise solution, some general properties of our system can still be deduced. In particular, as was mentioned before, counter-propagating Rossby waves allow for the development of baroclinic instability. Conversely, if Rossby waves are not propagating in opposite di-

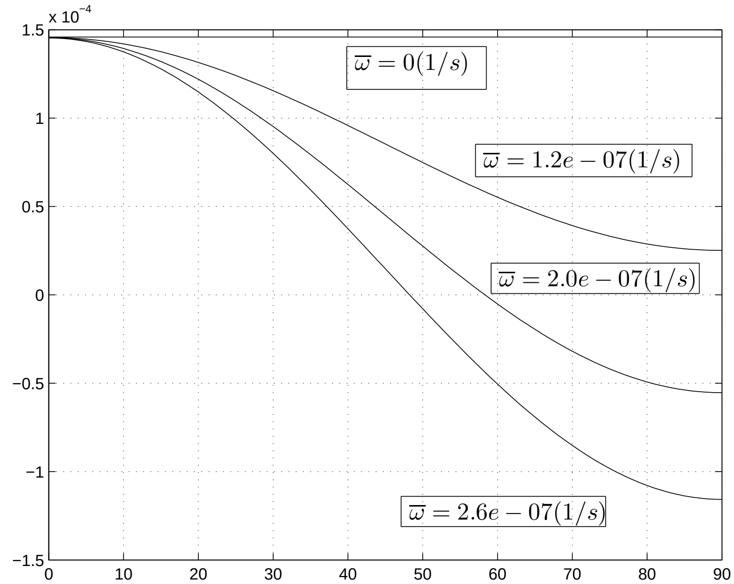

Figure 4.2: Meridional gradient of potential vorticity in the lower layer.

rections, we do not expect baroclinic instabilities to be present. As can be seen in the following equations, the direction of the Rossby waves depends critically on the angular velocity ($\bar{\omega}$), and the latitude ($\mu = \sin\phi$).

Using the previously assumed forms of $\bar{\psi}_1(\mu)$ and $\bar{\psi}_2(\mu)$ in the basic state version of (4.44) and (4.45), we obtain

$$\bar{q}_1(\mu) = 2\Omega\mu - 2\bar{\omega}\mu\left(1 + \frac{2(1+\gamma)\Omega^2 a^2 \mu^2}{g(1-\gamma)\bar{h}}\right), \tag{4.87}$$

$$\bar{q}_2(\mu) = 2\Omega\mu + 2\bar{\omega}\mu\left(1 + \frac{4\Omega^2 a^2 \mu^2}{g(1-\gamma)\bar{h}}\right), \tag{4.88}$$

so that the poleward gradients of the basic state potential vorticity are

$$\frac{d\bar{q}_1}{d\mu} = 2\Omega - 2\bar{\omega}\left(1 + \frac{6(1+\gamma)\Omega^2 a^2 \mu^2}{g(1-\gamma)\bar{h}}\right), \tag{4.89}$$

$$\frac{d\bar{q}_2}{d\mu} = 2\Omega + 2\bar{\omega}\left(1 + \frac{12\Omega^2 a^2 \mu^2}{g(1-\gamma)\bar{h}}\right). \tag{4.90}$$

Assuming $\bar{\omega} > 0$, it is clear that (4.90) will always be positive. Thus for counter-propagating Rossby waves to occur (4.89) will need to be negative. This depends on the specific value of $\bar{\omega}$ and the particular latitude most representative of the Rossby waves under consideration. The two given figures plot this dependence for several values of $\bar{\omega}$ at the equator. Figure 4.1 shows the vorticity 'parameter', which is the mean state, potential vorticity of the lower layer divided by the planetary vorticity. Figure 4.2 shows the meridional gradient of potential vorticity in the lower layer. When the curve for a given value of $\bar{\omega}$ becomes negative in figure 4.2 the corresponding latitude of the place where this transition occurs marks the point poleward of which counter-propagating Rossby-waves will be present.

Chapter 5

VERTICAL PROPAGATION OF STATIONARY PLANETARY WAVES

5.1 Introduction

Observations have shown that the majority of atmospheric energy is contained in the planetary-scale motions of the troposphere. Charney and Drazin (1961, CD61 hereafter) showed that throughout most of the year the upper stratospheric motions are largely independent of the motions in the lower atmosphere. This is somewhat surprising considering how much energy is contained in the troposphere. Why are the large amounts of energy in the lower atmosphere so inhibited from propagating into the stratosphere and interacting with the stratospheric general circulation?

Whether or not the planetary waves are able to propagate vertically has been observed to be highly dependent on the vertical structure of the zonal mean wind. Specifically, to quote from CD61, "Energy is trapped (reflected) in regions where the zonal winds are easterly or large and westerly." In other words, energy can only propagate vertically when the mean zonal winds are weak and westerly. The question of how and when planetary waves in the troposphere may propagate vertically then becomes a question of how the zonal mean wind interacts with these waves, and at what times the vertical structure of the wind is conducive for propagation.

Upper stratospheric observations have also shown that there is a circumpolar anticyclone during the summer and a circumpolar cyclone during the winter. We can conclude that due to the large zonal winds present in both of these circumstances, the tropospheric energy will be trapped or 'blocked' when these summer and winter circulations are active. It is during the times of transition between the summer and winter mean states of the upper atmosphere that vertical propagation of the planetary waves is possible.

Taroh Matsuno's work of 1970 (M70 hereafter) improved on the study of CD61 primarily in two ways. First, he generalized the mathematical analysis to include the lateral structure of waves in addition to the vertical structure, and second, a 'refractive index square' was computed and used as a tool to explain the spatial distribution of the propagating waves. In addition, Matsuno described the configuration of the refractive index square and the large zonal wind speeds in the upper stratosphere as analogous to a 'cavity' in the atmosphere. This cavity is then claimed to capture and amplify the waves which happen to propagate into it. Unfortunately, the mathematical derivation of the index of refraction was a bit awkward, and the result (his eq. 13) is clearly singular at the poles.

In the following, a brief review of Matsuno's paper is given to elucidate a few of the details regarding vertical propagation of stationary planetary waves. An alternative definition for the index of refraction will also be derived. This new result is a consequence of the governing equations presented in chapters 2 and 3 and results in a refractive index that is not singular at the poles. We also propose a straightforward physical mechanism describing the observed spatial distribution for the propagation of energy in the atmosphere.

5.2 Review of Matsuno 1970

5.2.1 Basic Equations

Matsuno's work assumed the dominant motions of interest to be in QG balance, adiabatic, frictionless, and slowly varying in time. The governing equations were then linearized by superimposing small-amplitude motions onto the basic state. Eliminating the vertical velocity between the thermodynamic (Matsuno's eq. 2 = M eq.2) and the vorticity equation (M eq.1) yielded the single equation governing the potential vorticity (M eq.3). This potential vorticity equation has three dependent variables: the geopotential (ϕ'), the perturbation velocity (v'), and the perturbation vorticity (ζ').

Using the geostrophic wind relationship to approximate relations between the geopotential, the meridional component of the perturbation velocity, and the perturbation vorticity, Matsuno obtained one equation with one unknown. This equation describes the vertical and lateral propagation of waves. Equation 10 of Matsuno's paper was then derived by separating out the longitudinal dependence of the governing equation and assuming an isothermal atmosphere. Concerning the assumption of an isothermal atmosphere, Matsuno writes, "this assumption may hold fairly well in the stratosphere but may not be good for the troposphere." M eq.9 is then re-arranged in such a way as to allow for (M eq.10) to contain a variable labeled as Q_m which is identified as the "refractive index square." This is the governing equation Matsuno solves throughout the rest of his paper. It is also important to note that these equations have been developed for stationary waves, so they represent waves that can be thought of as being 'glued' to the Earth at a given level.

5.2.2 The Basic State

For this study, Matsuno derived a wave equation depending on both latitude and height. To obtain a cross-section with a meridional and a vertical axis, the mean winds were zonally averaged. Due to questionable discrepancies in the observations, and the sensitivity of the differential equations to these discrepancies, combined with the physical constraints from the boundary conditions, Matsuno chose to use an idealized construction of the winter zonal winds rather than direct observations. This model of the zonal winds retained all of the major features of the observed wind system and is shown in Fig.5.1.A. Most of the differences between the constructed wind and the direct observations are near the equator. Because Matsuno's work focused primarily on the propagation of Rossby waves into the mid and upper latitudes he expected that the equatorial discrepancies in the wind field would not significantly alter the results.

The latitudinal gradient of potential vorticity of the basic state was then computed and is reproduced here as Fig. 5.1.B. The similarity between Figs. 5.1.A and 5.1.B indicates that the spatial distribution of the meridional gradient of PV is heavily dependent on the mean wind field. Matsuno concluded that the contribution by the mean wind was comparable to or larger than that made by β. This is an important point because β will not change with time, however, the mean wind field is ever-changing. Thus if the meridional gradient of PV is equally dependent on the basic wind

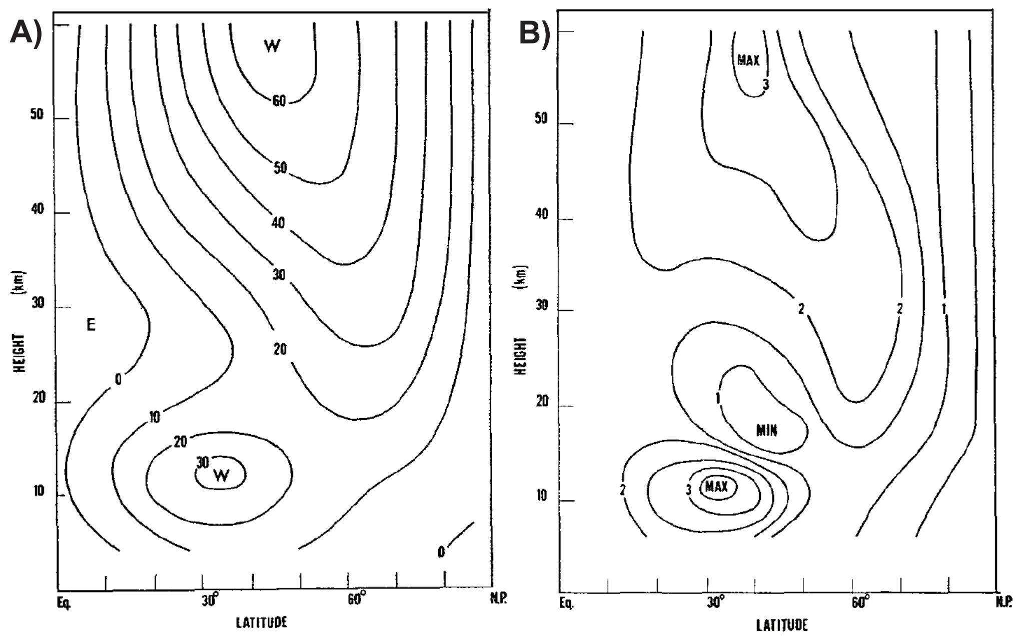

Figure 5.1: A. Approximate mean zonal winds. B. Latitudinal gradient of potential vorticity of the basic state. Both A and B from Matsuno (1970).

state and β we can expect that variations in the meridional gradient of PV will largely be due to the seasonally changing wind state.

The work of Charney and Stern (1962) showed that if a region of PV meridional gradient changes signs, then the possibility of wave instabilities in the polar night jet existed. A region of minimum PV gradient appears around $\sim 50°N$ at $\sim$ 20km in Fig. 5.1.B. It can be inferred from Fig. 5.1.A that this is due largely to the negative curvature of the wind profile in the latitude and vertical directions and the negative vertical shear in this region.

Due to the artificial wind model used to compute the gradient of PV, one may ask whether this region of minimum PV gradient is actually present in the atmosphere. Charney and Stern found that this configuration of the PV gradient is often met in the daily measurements of zonal wind. In addition, the work of Murakami (1965) verified the existence of negative values of the PV gradient in the vicinity of $50°N$ and 20 km. Thus observations have shown the structure of the PV gradient in the atmosphere to generally agree with the computations of Matsuno.

Next the refractive index square of the waves of zero-zonal wavenumber was computed. The result is shown in Matsuno's Fig. 3 and can be seen to generally decrease with increasing height and latitude. The arrangement of this refractive index square is credited with playing an important role in the propagation of waves.

5.3 Index of Refraction

Chapter 2 developed the governing prognostic equations for global quasi-geostrophic theory. For the purpose of comparing our system of equations to Matsuno's it is helpful to convert the equations of Chapter 2 to the standard log-pressure coordinate system, which is the vertical coordinate used in his study. The vertical coordinate in the log-pressure system is defined as $z \equiv H \ln(p_0/p)$, and the pseudo-density becomes $\rho = (p_0/g)e^{-z/H}$.

The only noticeable change this brings about in the nonlinear governing equations is to the invertibility principle where the new pseudo-density appears. Recall our system of governing equations:

$$\frac{\partial q}{\partial t} + \frac{1}{a^2}\frac{\partial(\psi, q)}{\partial(\lambda, \mu)} + \frac{2\Omega}{a^2}\frac{\partial \psi}{\partial \lambda} = 0 \tag{5.1}$$

for the interior of the domain and

$$\frac{\partial B}{\partial t} + \frac{1}{a^2}\frac{\partial(\psi, B)}{\partial(\lambda, \mu)} = 0, \tag{5.2}$$

for the lower boundary. The invertibility principle, including boundary conditions, is

$$q = \nabla^2\psi + 4\Omega^2\mu^2\, e^{z/H}\frac{\partial}{\partial z}\left(\frac{e^{-z/H}}{N^2}\frac{\partial \psi}{\partial z}\right), \tag{5.3}$$

$$\frac{\partial \psi}{\partial z} = 0 \quad \text{at } z = z_T, \qquad 4\Omega^2\mu^2\left(\frac{\partial \psi}{\partial z} - \frac{N^2}{g}\psi\right) = B \quad \text{at } z = 0. \tag{5.4}$$

The index of refraction is now derived from our set of governing equations. We begin by using the same method of linearization that was employed in Chapter 4 to linearize (5.1) which yields

$$\left(\frac{\partial}{\partial t} + \bar{\omega}\frac{\partial}{\partial \lambda}\right)q' + \frac{\partial \bar{q}}{\partial \mu}\frac{\partial \psi'}{a^2\partial \lambda} + \frac{2\Omega}{a^2}\frac{\partial \psi}{\partial \lambda} = 0, \tag{5.5}$$

where $\bar{\omega}(\phi, z)$ is the angular velocity of the basic zonal flow, and

$$q' = \nabla^2\psi' + 4\Omega^2\mu^2\, e^{z/H}\frac{\partial}{\partial z}\left(\frac{e^{-z/H}}{N^2}\frac{\partial \psi'}{\partial z}\right), \tag{5.6}$$

$$\bar{q}(\mu, z) = -\frac{\partial[\bar{\omega}(1-\mu^2)]}{\partial \mu} + 4\Omega^2\mu^2\, e^{z/H}\frac{\partial}{\partial z}\left(\frac{e^{-z/H}}{N^2}\frac{\partial \bar{\psi}}{\partial z}\right). \tag{5.7}$$

Considering stationary waves and trying a solution to (5.5) of the form

$$\psi'(\lambda, \mu, z) = e^{z/2H}\Psi_m(\mu, z)e^{im\lambda} \tag{5.8}$$

we obtain

$$\begin{aligned}\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial \Psi_m}{\partial \mu}\right) - \frac{m^2}{1-\mu^2}\Psi_m + 4\Omega^2 a^2\mu^2 e^{z/2H}\frac{\partial}{\partial z}\left(\frac{e^{-z/H}}{N^2}\frac{\partial(e^{z/2H}\Psi_m)}{\partial z}\right)\\ + \frac{1}{\bar{\omega}}\left(2\Omega + \frac{\partial \bar{q}}{\partial \mu}\right)\Psi_m = 0.\end{aligned} \tag{5.9}$$

Note that in (5.9) the longitudinal dependence has been 'separated out' of (5.5) due to the form of solution that was chosen. The above equation can be rewritten as

$$\mathcal{L}_m\Psi_m + \frac{1}{\bar{\omega}}\left(2\Omega + \frac{\partial \bar{q}}{\partial \mu}\right)\Psi_m = 0, \tag{5.10}$$

where

$$\mathcal{L}_m\Psi_m = \frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial \Psi_m}{\partial \mu}\right) - \frac{m^2}{1-\mu^2}\Psi_m + 4\Omega^2 a^2\mu^2 e^{z/2H}\frac{\partial}{\partial z}\left(\frac{e^{-z/H}}{N^2}\frac{\partial(e^{z/2H}\Psi_m)}{\partial z}\right). \tag{5.11}$$

Now the index of refraction is chosen to be $\frac{1}{\bar{\omega}}\left(2\Omega + \frac{\partial \bar{q}}{\partial \mu}\right)$. As can be seen, defining the index of refraction in this way is a comparatively simple process. Another important point to be made is that the perturbation potential vorticity (5.6) has been kept together in the same term. As will be clear in the following, this term is split up in Matsuno's derivation. The significance of this alternate definition for the index of refraction is that the observations of wave propagation can be explained simply in terms of the potential vorticity and the spatial distribution of the zonal wind.

Matsuno assumed the atmosphere to be isothermal which implies that N^2 can be replaced by an analogous N^2 for an isothermal atmosphere having a scale height of H ($H = RT_0/g$ where R is the gas constant). He then took $-m^2/(1-\mu^2)$ (which comes from the latitudinal piece of the Laplacian operator in the potential vorticity) and combined it with one of the pieces from the $\partial/\partial z$ term in (5.6) when it is expanded via the product rule. These two pieces, plus the meridional gradient of mean state potential vorticity, make up Matsuno's index of refraction. In his equation 11 he thus writes the index of refraction as

$$Q_m = \frac{1}{\bar{\omega}} \frac{\partial \bar{q}}{\cos\phi \partial \phi} - \frac{m^2}{\cos^2\phi} - \frac{\Omega^2 a^2 \sin^2\phi}{N^2 H^2}. \tag{5.12}$$

where the term $\partial\bar{q}/\cos\phi\partial\phi$ is a slightly different expression than has been defined via (5.7) due to differences of the initial approximations. Note the second term of (5.12) will be singular at the poles. In the following section the mathematical differences and similarities between Matsuno's work and the present work will be shown.

5.3.1 Analysis of Governing Equation

The previous section on Matsuno's basic equations outlines his derivation of a single equation in a single unknown. The unknown (Φ_d) represents the deviation of the geopotential field from its mean reference state. For the purpose of comparison, this equation (M eq. 9) is reproduced here:

$$\bar{\omega}\frac{\partial}{\partial\lambda}\left[\frac{\sin^2\phi}{\cos\phi}\frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{\sin^2\phi}\frac{\partial\Phi_d}{\partial\phi}\right) + \frac{\partial^2\Phi_d}{\cos^2\phi\partial\lambda^2} + 4\Omega^2a^2\sin^2\phi\frac{\partial}{p\partial z}\left(\frac{p}{N^2}\frac{\partial\Phi_d}{\partial z}\right)\right] + \frac{\partial\bar{q}}{\partial\phi}\frac{\partial\Phi_d}{\cos\phi\partial\lambda} = 0. \tag{5.13}$$

Matsuno used θ as his meridional coordinate and ϕ' as representing the deviation of the geopotential field from its mean state, here θ has been switched to ϕ and ϕ' to Φ_d to maintain consistency with the rest of this present work.

We desire to compare our system of equations with (5.13). To do this we begin with (5.5). Matsuno assumed a stationary state ($\partial/\partial t = 0$), and so we do the same. This leads to

$$\bar{\omega}\frac{\partial q'}{\partial\lambda} + \frac{\partial\bar{q}}{\partial\mu}\frac{\partial\psi'}{a^2\partial\lambda} + \frac{2\Omega}{a^2}\frac{\partial\psi'}{\partial\lambda} = 0. \tag{5.14}$$

Now q' from (5.6) is substituted into (5.14) and the result is written in terms of p rather than $e^{z/H}$.

$$\bar{\omega}\frac{\partial}{\partial\lambda}\left[\nabla^2\psi' + 4\Omega^2\mu^2\frac{\partial}{p\partial z}\left(\frac{p}{HN^2}\frac{\partial\psi'}{\partial z}\right)\right] + \left(2\Omega + \frac{\partial\bar{q}}{\partial\mu}\right)\frac{\partial\psi'}{a^2\partial\lambda} = 0. \tag{5.15}$$

Again using the balance relation $\Phi_d = 2\Omega\mu\psi'$, defined in Chapter 3, the horizontal Laplacian operator can be expanded in the following way

$$\nabla^2\psi' = \frac{1}{2\Omega\mu a^2}\left[\frac{\partial^2\Phi_d}{(1-\mu^2)\partial\lambda^2} + \frac{\sin^2\phi}{\cos\phi}\frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{\sin^2\phi}\frac{\partial\Phi_d}{\partial\phi}\right) + \frac{2\Phi_d}{\mu^2}\right]. \tag{5.16}$$

Substitution of this into (5.15) and transforming each of the terms in (5.15) from the streamfunction (ψ) to the geopotential (Φ_d) yields

$$\begin{aligned}\frac{1}{2\Omega\mu a^2}\bar{\omega}\frac{\partial}{\partial\lambda}\left[\frac{\partial^2\Phi_d}{(1-\mu^2)\partial\lambda^2} + \frac{\sin^2\phi}{\cos\phi}\frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{\sin^2\phi}\frac{\partial\Phi_d}{\partial\phi}\right) + \frac{2\Phi_d}{\mu^2} + 4\Omega^2a^2\mu^2\frac{\partial}{p\partial z}\left(\frac{p}{HN^2}\frac{\partial\Phi_d}{\partial z}\right)\right]\\ + \frac{1}{2\Omega\mu a^2}\left(2\Omega + \frac{\partial\bar{q}}{\partial\mu}\right)\frac{\partial\Phi_d}{\partial\lambda} = 0.\end{aligned} \tag{5.17}$$

Upon simplification we get

$$\begin{aligned}\bar{\omega}\frac{\partial}{\partial\lambda}\left[\frac{\sin^2\phi}{\cos\phi}\frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{\sin^2\phi}\frac{\partial\Phi_d}{\partial\phi}\right) + \frac{\partial^2\Phi_d}{\cos^2\phi\partial\lambda^2} + \frac{2\Phi_d}{\sin^2\phi} + 4\Omega^2a^2\sin^2\phi\frac{\partial}{p\partial z}\left(\frac{p}{HN^2}\frac{\partial\Phi_d}{\partial z}\right)\right]\\ + \left(2\Omega\cos\phi + \frac{\partial\bar{q}}{\partial\cos\phi}\right)\frac{\partial\Phi_d}{\cos\phi\partial\lambda} = 0.\end{aligned} \tag{5.18}$$

There are two apparent differences, other than the differences between the index of refraction, between the above equation and (5.13); the term $(2\Phi_d/\sin^2\theta)$ and the H term in the denominator of the $\frac{\partial}{\partial z}$ term of (5.18). In Matsuno's equation (5.13) an isothermal atmosphere had not yet been assumed. As mentioned in the previous section, when this assumption is made, the static stability term N^2 is replaced by a constant N^2. At that point, the scale height is introduced. So the appearance of H in (5.18) does not represent a significant difference between (5.13) and (5.18). When approximating the perturbation velocity, Matsuno included an ageostrophic term to try and maintain consistency in the energy equation. Because this was not a focus of the present chapter it was not felt necessary to make this approximation and the pure geostrophic system was used. It is suspected that this difference is what accounts for the $(2\Phi_d/\sin^2\theta)$ term in (5.18). In any case, this discrepancy does not significantly impact the definition of the index of refraction or the basic dynamics of the vertically propagating planetary waves.

5.4 Chapter Summary

The governing equation for vertically propagating planetary waves has been formulated in a global coordinate system. The conditions necessary for the waves to propagate are outlined. The meridional gradient of the PV provides the restoring mechanism for these waves by resisting meridional displacements.

A critical review of Matsuno (1970) is given with an emphasis on reformulating what he termed, “the refractive index square (Q_m)”. It is shown that $\frac{1}{\bar{\omega}}\left(2\Omega + \frac{\partial \bar{q}}{\partial \mu}\right)$ is a more natural definition for the effective index of refraction than Matsuno’s eq. (11). In defining the index of refraction this way, Matsuno split up the horizontal Laplacian operator with one term being included in Q_m while the other term was not. The $\partial/\partial z$ is also split in two with one piece being included in Q_m. The affect of this is a Q_m that is singular at the poles and a confusing equation (M eq. 11). If Q_m is defined without these two pieces, the singularity is no longer present and the resulting equation is organized in a more concise and physically relevant quantity given by equation (5.10). This equation leads to a natural definition of an effective index of refraction given by quantity $\frac{1}{\bar{\omega}}\left(2\Omega + \frac{\partial \bar{q}}{\partial \mu}\right)$ which physically represents the meridional gradient of potential vorticity, divided by the angular velocity.

Matsuno’s Fig. 2 shows a cross-section of $\frac{\partial \bar{q}}{\partial \phi}$ and is reproduced here as Fig. 5.1.B. This figure is extremely useful when $\frac{\partial \bar{q}}{\partial \phi}$ is understood to provide the restoring force for the Rossby waves. The prominent minimum of $\frac{\partial \bar{q}}{\partial \phi}$ at roughly 18 km and $40N$ explains why we do not see planetary waves propagating upward until they are north of this minimum. Once the waves propagate to around 20 km, they encounter high zonal winds. These high winds are not conducive for the waves propagating to a higher altitude. The planetary waves then tend to move somewhat equatorward.

The effective index of refraction defined in this chapter offers a clear physical explanation for the observed propagation of stationary planetary waves, namely that the meridional gradient of potential vorticity, divided by the angular velocity acts as the restoring force to the planetary waves. Thus we can conclude that the regions containing minimal values of the meridional gradient of potential vorticity will not be conducive to the propagation of these waves. In other words, computing the index of refraction as defined by Matsuno is unnecessary, one need only look at the distribution of the meridional gradient of potential vorticity divided by the angular velocity to arrive at the same conclusions concerning the likely locations for vertical propagation to occur.

It is clear that the terminology and definition concerning the ‘index of refraction’ given by Matsuno (1970) has been largely adopted in the literature. The use of this quantity has remained similar as many authors have tried to use it as a diagnostic to describe the behavior of large-scale waves forced in the troposphere and their propagation into the upper troposphere and stratosphere. Basically, where the index of refraction is a minimum, the propagation of planetary waves is assumed to be unlikely, while a region of large index of refraction allows for increased propagation of waves.

A brief review of a few relevant papers has shown that Lin (1982), Chen and Robinson (1992), Hu and Tung (2001), and Li et al. (2006) have all defined and used an index of refraction that is analogous in form to the index defined by Matsuno (1970), while O’Neill and Youngblut (1982) (OY82 hereafter) have defined an index that is very similar to the one derived in this present work. To distinguish between these two forms of the index, the index derived in this chapter and OY82 is referred to as the ‘effective’ index of refraction, and the index derived Matsuno is referred to as the index of refraction.

It is quite interesting that the work of OY82 did not change the standard definition of the index of refraction. Although their work was published nearly 25 years ago, Matsuno’s form of the index still remains the standard form used in the literature. It should also be pointed out that the form of index derived in OY82 was derived using ray theory, yet they arrived at the same expression produced in this work, where we have simply linearized our theory and substituted in the appropriate wave solution. It is encouraging to see that the expression derived in this chapter can be derived in multiple ways.

Although an index analogous to Matsuno’s was used in the work of Hu and Tung (2001), their figure 5 proves useful to our argument. In Figs 5c and 5d they plot the index of refraction, while in Figs. 5e and 5f they plot the meridional gradient of potential vorticity divided by the angular velocity. The two figures are very similar. Our argument that the same conclusions can be drawn from both figures and therefore that Figs. 5c and 5d are unnecessary seems to be supported by this figure.

Matsuno's research was excellent work and provided many helpful insights to this topic. This chapter simply reformulates and slightly extends his work in an attempt to further clarify and generalize the phenomenon of energy propagation into the stratosphere.

Chapter 6

GEOSTROPHIC TURBULENCE ON THE SPHERE

6.1 Introduction

Turbulence is ubiquitous in nature. It leads to some of the most interesting phenomena while continuously evading thorough understanding. Observations make it clear that turbulence is a dominant feature of the atmosphere, therefore furthering our understanding of the relationship between turbulent fluids and the large-scale atmospheric motions is important. Rick Salmon defines geostrophic turbulence as, "strongly nonlinear, rapidly rotating, stably stratified flow." (Lectures on G.F.D, p. 263) Using the balanced model that is presented in this thesis to model geostrophic turbulence in a fully spherical and rotating domain is a much anticipated direction for future research.

Peter Rhines was the first to investigate the affects rotation has on two-dimensional turbulence (1975). His studies led to the definition of the "Rhines scale" on the β-plane. Vallis and Maltrud (1993) derived an anisotropic Rhines scale on the β-plane. Huang and Robinson (1998) then gave a spherical version. This chapter further generalizes the Rhines scale to the continuously stratified sphere and derives the Parseval relation for both the energy and the potential enstrophy.

The Rhines scale is a useful tool for conceptualizing the physical properties of a flow. When fluids are dominated by motions which can be approximated as two-dimensional, the well known upscale energy cascade dominates the flow of energy. As this takes place, the characteristic scale of the fluid increases. Once the Rhines scale is reached, the upscale energy cascade ceases and the flow becomes dominated by wave motions. We can thus use this parameter to visualize the critical scale dividing turbulent from wave-like motion. Planetary rotation effectively puts an upper limit on the upscale transfer of energy.

Comparison of the stationary terms in the governing equation leads directly to the Rhines scale. The non-linear term gives rise to turbulent motions in the fluid, while the beta-term is the source of wave motions. The Rhines scale results when the ratio of these two terms is approximately unity. It is important to note that the beta-term is zero for the zonal modes. This is the source of the anisotropic Rhines scale and implies the upscale transfer of energy will not be inhibited for motions in the zonal direction.

6.2 Generalized Rhines Length

Recall the governing equation derived in chapter 2

$$\frac{\partial q}{\partial t} + \frac{1}{a^2}\frac{\partial(\psi, q)}{\partial(\lambda, \mu)} + \frac{2\Omega}{a^2}\frac{\partial \psi}{\partial \lambda} = 0, \tag{6.1}$$

and the invertibility principle derived in chapter 3

$$\nabla^2 \psi_\ell - \frac{\epsilon_\ell \mu^2}{a^2}\psi_\ell = q_\ell, \tag{6.2}$$

where the subscript ℓ designates vertical modes.

As can be seen, the PV conservation relation (6.1) is nonlinear and thus proves difficult to solve. In contrast, the invertibility principle can be analytically solved through the use of spheroidal harmonics. Prediction of q can then be made with pseudospectral methods.

Spheroidal harmonics $\mathcal{S}_{mn}(\epsilon_\ell; \lambda, \mu)$ satisfy

$$\nabla^2 \mathcal{S}_{mn} - \frac{\epsilon_\ell \mu^2}{a^2}\mathcal{S}_{mn} = -\frac{\alpha_{mn}(\epsilon_\ell)}{a^2}\mathcal{S}_{mn}, \tag{6.3}$$

where $-\alpha_{mn}(\epsilon_\ell)/a^2$ is the eigenvalue. Spheroidal harmonics have the separable form $\mathcal{S}_{mn}(\epsilon_\ell; \lambda, \mu) = S_{mn}(\epsilon_\ell; \mu)e^{im\lambda}$, which, when substituted in (6.3), yields the ordinary differential equation

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dS_{mn}}{d\mu}\right] + \left(\alpha_{mn}(\epsilon_\ell) - \epsilon_\ell \mu^2 - \frac{m^2}{1-\mu^2}\right)S_{mn} = 0 \tag{6.4}$$

for the meridional structure $S_{mn}(\epsilon_\ell; \mu)$, which is often referred to as the “spheroidal wave function.”

The orthonormality relation for spheroidal harmonics is

$$\frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\mathcal{S}_{mn}(\epsilon_\ell;\lambda,\mu)\mathcal{S}^*_{m'n'}(\epsilon_\ell;\lambda,\mu)\,d\lambda\,d\mu = \begin{cases}1 & (m',n')=(m,n)\\ \\ 0 & (m',n')\neq(m,n),\end{cases} \tag{6.5}$$

or, in terms of $S_{mn}(\epsilon_\ell;\mu)$,

$$\frac{1}{2}\int_{-1}^{1}S_{mn}(\epsilon_\ell;\mu)S_{mn'}(\epsilon_\ell;\mu)\,d\mu = \begin{cases}1 & n'=n\\ \\ 0 & n'\neq n.\end{cases} \tag{6.6}$$

The spheroidal harmonic transform pair for the potential vorticity is

$$q_{\ell mn}(t) = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} q_\ell(\lambda,\mu,t)\,\mathcal{S}^*_{mn}(\epsilon_\ell;\lambda,\mu)\,d\lambda\,d\mu, \tag{6.7}$$

$$q_\ell(\lambda,\mu,t) = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty} q_{\ell mn}(t)\,\mathcal{S}_{mn}(\epsilon_\ell;\lambda,\mu). \tag{6.8}$$

An identical transform pair exists for $\psi_{\ell mn}(t)$ and $\psi_\ell(\lambda,\mu,t)$. Note that (6.7) can be obtained by multiplying (6.8) by $\mathcal{S}^*_{m'n'}(\epsilon_\ell);\lambda,\mu$, integrating over the sphere, and using the orthonormality relation (6.5).

The total energy principle can be derived by multiplying (6.1) by $-\psi$, performing a vertical transform and integrating over the sphere. The result is

$$\frac{dE}{dt} = 0, \tag{6.9}$$

where each vertical mode of E is given by

$$E_\ell = \frac{1}{8\pi}\int_{-1}^{1}\int_{0}^{2\pi}\left(\nabla\psi_\ell\cdot\nabla\psi_\ell + \frac{\epsilon_\ell\mu^2}{a^2}\psi_\ell^2\right)d\lambda\,d\mu = -\frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\tfrac{1}{2}\psi_\ell q_\ell\,d\lambda d\mu. \tag{6.10}$$

The total energy can be found by

$$E = \sum_{\ell=0}^{\infty}E_\ell. \tag{6.11}$$

Using the spheroidal harmonic expansions for $\psi_\ell(\lambda,\mu,t)$ and $q_\ell(\lambda,\mu,t)$, we can express E in spectral space as

$$
\begin{aligned}
E_\ell &= -\frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} \tfrac{1}{2}\left(\sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}\psi_{\ell mn}\mathcal{S}_{mn}\right)\left(\sum_{m'=-\infty}^{\infty}\sum_{n'=|m'|}^{\infty}q^*_{\ell m'n'}\mathcal{S}^*_{m'n'}\right)d\lambda d\mu \\
&= -\sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}\sum_{m'=-\infty}^{\infty}\sum_{n'=|m'|}^{\infty}\tfrac{1}{2}\psi_{\ell mn}q^*_{\ell m'n'}\left(\frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\mathcal{S}_{mn}\mathcal{S}^*_{m'n'}d\lambda d\mu\right) \\
&= -\sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}\tfrac{1}{2}\psi_{\ell mn}q^*_{\ell mn} = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}\frac{\alpha_{mn}(\epsilon_\ell)}{2a^2}\left|\psi_{\ell mn}\right|^2 = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}E_{\ell mn}, \quad (6.12)
\end{aligned}
$$

where we have used the orthonormality relation (6.5) to obtain the first equality in the last line, the invertibility principle $q^*_{\ell mn} = -a^{-2}\alpha_{mn}(\epsilon_\ell)\psi^*_{\ell mn}$ to obtain the second equality in the last line, and where we have defined

$$E_{\ell mn} = \tfrac{1}{2}\frac{\alpha_{mn}(\epsilon_\ell)}{a^2}\left|\psi_{\ell mn}\right|^2. \quad (6.13)$$

Equation (6.12) is the “Parseval relation” for the total energy. It allows us to compute the total energy of the atmosphere either by an integral of $\frac{1}{2}(\nabla\psi_\ell\cdot\nabla\psi_\ell + \epsilon_\ell a^{-2}\mu^2\psi_\ell^2)$ over physical space or by a sum of $\frac{1}{2}a^{-2}\alpha_{mn}(\epsilon_\ell)\left|\psi_{\ell mn}\right|^2$ over spheroidal harmonic wavenumber space.

The potential enstrophy principle, obtained by multiplying (6.1) by q and integrating over the sphere, is

$$\frac{dZ}{dt} = 0, \quad (6.14)$$

where Z is the potential enstrophy. Each normal mode of the potential enstrophy can be obtained from

$$Z_\ell = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\tfrac{1}{2}q_\ell^2\, d\lambda d\mu. \quad (6.15)$$

Using the spheroidal harmonic expansion for $q_\ell(\lambda,\mu,t)$, we can express Z in spectral space as

$$
\begin{aligned}
Z_\ell &= \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} \tfrac{1}{2}\left(\sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}q_{\ell mn}\mathcal{S}_{mn}\right)\left(\sum_{m'=-\infty}^{\infty}\sum_{n'=|m'|}^{\infty}q^*_{\ell' m'n'}\mathcal{S}^*_{m'n'}\right)d\lambda d\mu \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}\sum_{m'=-\infty}^{\infty}\sum_{n'=|m'|}^{\infty}\tfrac{1}{2}q_{\ell mn}q^*_{\ell' m'n'}\left(\frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\mathcal{S}_{mn}\mathcal{S}^*_{m'n'}d\lambda d\mu\right) \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}\tfrac{1}{2}q_{\ell mn}q^*_{\ell mn} = \sum_{m=-\infty}^{\infty}\sum_{n=|m|}^{\infty}Z_{\ell mn}, \quad (6.16)
\end{aligned}
$$

where we have again used the orthonormality relation (6.5) to obtain the last line in (6.16). Using the invertibility principle and the definition of $E_{\ell mn}$, this can also be written as

$$Z_{\ell mn} = \tfrac{1}{2} \left| q_{\ell mn} \right|^2 = \frac{\alpha_{mn}(\epsilon_\ell)}{a^2} E_{\ell mn}. \tag{6.17}$$

Equation (6.16) is the "Parseval relation" for the potential enstrophy. It allows us to compute the total potential enstrophy either by an integral of $\frac{1}{2} q_\ell^2$ over physical space or by a sum of $\frac{1}{2} \left| q_{\ell mn} \right|^2$ over spheroidal harmonic wavenumber space.

As discussed by Charney (1971), energy moves toward lower wavenumber while potential enstrophy moves toward higher wavenumber. Because of this cascade in opposite directions, potential enstrophy is more subject to small scale dissipation, and hence can be selectively decayed while energy is nearly conserved even in the presence of dissipation. This is the essence of the selective decay hypothesis.

One of the uses of this numerical model is to study geostrophic turbulence on the sphere. Equation (6.1) contains nonlinear advection of q and a linear term associated with Rossby-Haurwitz waves. The Rossby-Haurwitz wave frequency is given by $2\Omega m/\alpha_{mn}(\epsilon_\ell)$. The turbulent frequency is given by $a^{-1} \left[\alpha_{mn}(\epsilon_\ell)\right]^{\frac{1}{2}} V_{\text{rms}}$, where V_{rms} is the root-mean-square velocity. The dynamics is wavelike if $2\Omega m/\alpha_{mn}(\epsilon_\ell) \gg a^{-1} \left[\alpha_{mn}(\epsilon_\ell)\right]^{\frac{1}{2}} V_{\text{rms}}$, while it is dominated by turbulence if $2\Omega m/\alpha_{mn}(\epsilon_\ell) \ll a^{-1} \left[\alpha_{mn}(\epsilon_\ell)\right]^{\frac{1}{2}} V_{\text{rms}}$. The anisotropic Rhines barrier is defined by equating the two time scales, which, after some rearrangement, can be written as

$$\frac{m}{\left[\alpha_{mn}(\epsilon_\ell)\right]^{\frac{3}{2}}} = \frac{V_{\text{rms}}}{2\Omega a}. \tag{6.18}$$

For a given $V_{\text{rms}}/(2\Omega a)$, (6.18) defines a curve in the spheroidal harmonic wavenumber plane (m, n). As mentioned previously, for flows with the energy at scales smaller than the scale defined by this curve, the motion will be turbulent. For flows with the energy at scales larger than the curve, the motion will be wave-like.

Chapter 7

CONCLUSIONS

This work has successfully developed a globally applicable balanced model. Two of the primary advantages this system of equations offers is the lack of a singularity at the equator and the ability to model inter-hemispheric motions. Traditionally, quasi-geostrophic models have been thought to be necessarily non-global due to the break-down of the equations at the equator. The primary theoretical difference between the balanced system presented here and previous formulations of quasi-geostrophic theory rests in the choice of which predictive variable to use. Here the streamfunction field is used, whereas previously the geopotential field was the variable of choice. In contrast with the geostrophic wind, the streamfunction is a well-defined field over the entire globe.

As to which of the two variables (ψ, Φ) most accurately predicts the evolution of a particular system –that depends on whether the temperature field or the wind field is more important to the prediction of a particular phenomenon. Phillips (1998) gave an excellent review of the predictive differences between ψ and Φ in his analysis of the November 24, 1950 storm. His support for the use of the streamfunction as the predictive variable in the quasi-geostrophic system gives confidence for this present work in spite of the popularity of using the geopotential field.

Chapter 4 successfully generalized the Charney-Stern necessary condition for combined barotropic-baroclinic instability. This generalization was accomplished using two distinct physical representations of the flow. The first method follows Eliassen (1983) and quantifies the meridional motion of a fluid parcel. Simply implementing this motion into the quasi-balanced equations leads to the integral relationships that imply the necessary condition for instability of the flow. The second method follows Charney and Stern (1962) by assuming an exponential wave form solution ($\psi' = \Psi e^{i(m\lambda - \nu t)}$) to the quasi-balanced equations presented in chapters 2 and 3. Plugging this solution into the governing equation and then integrating over the domain leads to the same necessary condition for instability. The successful derivation of these conditions is not meant to represent a profound new principle governing fluid flows, but rather, provides verification that the particular form of spherical quasi-balanced theory presented in this work reproduces well known phenomena in fluids.

The last section of chapter 4 examines the quasi-geostrophic two-layer model. Unfortunately, analytic solutions to this model were not found. Numerous attempts were made to solve these equations including a normal-mode transform and a Boussinesq-like approximation, but each attempt resulted in further complications. Despite the lack of analytic solutions, it is possible to solve the system through numerical techniques. For a two-layer model, counter-propagating Rossby waves must be present for baroclinic instability to be generated. This places a constraint on the possible values of $\bar{\omega}$ for a baroclinically unstable environment to be possible. Figures 4.1 and 4.2 plot various values of $\bar{\omega}$ as a function of latitude. These figures show various regimes where instability is possible.

In chapter 5, the well known phenomenon of the vertical propagation of energy into the stratosphere is examined. The primary result is the modification of the index of refraction defined by Matsuno (1970). Matsuno's definition was unrealistic in that his index of refraction was singular at the poles. The mathematical method by which he arrived at this result seemed awkward. With the definition for the effective index of refraction given in chapter 5 we see that the propagation of energy due to stationary planetary waves can be explained with the observed spatial distribution of the meridional gradient of potential vorticity divided by the angular velocity.

Implications of this spherical quasi-balanced model to geostrophic turbulence are briefly discussed in chapter 6. Specifically, the well-known Rhines scale is further generalized to an anisotropic barrier in three dimensional wavenumber space. When the dominant scales of motion are below this critical value, the flow exhibits turbulent behavior, when the dominant scales of motion reach the Rhines scale and beyond, the flow is dominated by waves. This has important implications for the modeling of features such as zonal jets in the general circulation.

Although this system eliminates singularities at the equator, it is clearly not perfect. The geopotential, as mathematically represented in this work ($\Phi_d = 2\Omega\mu\psi$) must equal zero at the equator. From a physical standpoint, this is obviously erroneous; we know the geopotential does not equal zero in the equatorial regions. This turns out not to be such a huge problem when the streamfunction is the primary variable we are interested in because there are no restrictions on its value at the equator. It is through this streamfunction that the present model can include interaction between the hemispheres. This inter-hemispheric interaction is a significant improvement over previous QG models that often used the equator as nothing more than a boundary condition. An additional weakness of the theory presented here is its inability to accurately simulate flows near regions of convection. This is because we have assumed the dominant flow to be non-divergent, so that regions of strong vertical motions will fall outside of the domain of this model. It should also be kept in mind that this model has been developed only for flows with a small Rossby number and thus can only simulate large scale flows.

It has been shown (Schubert and Masarik, 2006) that the zero value of Φ at the equator only distorts the overall geopotential field to within a few degrees latitude of the equator. Thus the geopotential field can still provide a useful tool in our balanced model, if it is kept in mind that the values near the equator are incorrect.

The purpose of this work has been to begin the process, and lay as much of a foundation as possible for the development of a mathematically sound large scale system of balanced equations. The goal of Chapters 4 through 6 was to demonstrate that this global quasi-balanced model succeeds in reproducing and generalizing some of the well-known behaviors that have traditionally been studied on the f-plane, β-plane, or non-global spherical models. Theoretically, this model should be capable of generalizing most of the problems that wave been studied in the previous limited domains, not just the three phenomena examined with this research.

This sets the stage for future work on many problems. The most obvious avenue for future work is to implement the theory presented here in a series of numerical modeling experiments. This could include further work on vertical propagation of energy into the stratosphere and experiments related to baroclinic instability on the sphere. If sufficiently developed, numerical experiments of geostrophic turbulence could shed light on such things as the formation of jets in planetary atmospheres. Work is currently being done to compute the necessary eigenfunctions for the spheroidal harmonics. This will allow computation of the ‘Rhines-surface’ in three-dimensional wavenumber space. To the best of my knowledge, the Rhines-length has previously only been applied to two-dimensional surfaces. Further generalizing this quasi-balanced model to encompass flows at a large Rossby number (analogous to the semi-geostrophic approximation) is another possible route for future research.

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