

Quasi-balanced Dynamics in the Tropics

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(Manuscript received 19 April 1989, in final form 28 February 1990)

ABSTRACT

We propose a quasi-balance dynamical system of equations for slowly evolving, large-scale motions in the tropics. Unlike other balance schemes, equatorially trapped Kelvin waves are included. Following the lead of Gill, this system is based on the neglect of the meridional acceleration (the so-called long-wave approximation).

Consistent budgets for kinetic energy and potential vorticity are derived, with those quantities appropriately redefined. Unlike the quasi-geostrophic system, vertical advection of potential vorticity as well as of potential temperature is relevant. A primary circulation is defined, involving the zonal wind, geopotential, and potential temperature, such that potential vorticity depends on latitudinal and vertical gradients of the primary circulation. A residual secondary circulation is governed by an elliptic equation for a streamfunction in the meridional plane. The invertibility principle is affected by the Kelvin waves with zero potential vorticity.

The eigenfrequencies for a tropospheric first internal mode are obtained both for the primitive equations and the quasi-balanced system. Applications of this dynamical system are proposed for diagnostic analysis and for investigating the slow evolution of the large-scale flow in the tropics.

1. Introduction

Quasi-geostrophic theory provides a dynamical framework for understanding the slowly evolving, meteorologically significant large-scale phenomena in middle latitudes. Within this approximation the faster gravity waves are filtered out by neglect of the divergent component of the horizontal flow, which is typically smaller than the rotational component by a factor of the Rossby number.

Balanced dynamical theories have been proposed for the tropics. Both linear balance and nonlinear balance approximations have been used to filter out the faster divergent motions by neglecting the time rate of change of divergence in the divergence budget. These approximations suffer from both a practical and a dynamical disadvantage. An elliptic equation must be solved to obtain the rotational wind from the mass field; in practice, there can be significant regions of the tropics where the ellipticity condition is not satisfied (Paegle and Paegle 1976) for the slow motions. Dynamically, a class of slow motions exists in the tropics that are balanced in the meridional component of the momentum equation but act as gravity waves in the zonal component;

these equatorial Kelvin waves are excluded from the balance approximations (Moura 1976). Equatorial Kelvin waves have been postulated as a description of wave phenomena in the stratosphere (Wallace and Kousky 1968), in the troposphere (Wang 1988), and in the ocean (Busalacchi and O'Brien 1981). As demonstrated by Haltiner and Williams (1980, p. 69), the usual balance equations formulated in isobaric coordinates suffer from two additional deficiencies: namely, the lack of a simple potential vorticity conservation equation and the lack of a simple equation for the secondary circulation, as the omega equation provides for the ageostrophic motions in quasi-geostrophic theory.

Our search for a balanced dynamical system applicable to the tropics is guided by observations. Wallace (1983) noted that the zonal component of the stationary wave kinetic energy dominates the meridional component from the deep tropics to 45°N, and ascribed this structure to the fact that the meridional scale of the waves is much shorter than the zonal scale. Wallace and Blackmon (1983) pointed out a similar relationship for low-frequency fluctuations, with periods longer than 10 days, especially in lower latitudes. Hence the slowly evolving circulations in the tropics, with zonal velocities dominating over meridional velocities, are intimately related to the anisotropy of horizontal scales for the geopotential field.

In this paper we propose a balance scheme for tropical motions that includes Kelvin waves as well as long Rossby waves. It is based fundamentally on the empirically derived assumption that for slowly evolving, large-scale motions in the tropics (and also the extratropics, for that matter), the meridional pressure-gra-

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dient force balances the Coriolis force. In section 2 we show that this balance in the meridional momentum equation can be obtained from a consideration of the small ratio between meridional and zonal length scales. Although similar to the quasi-geostrophic approximation the distinction of this quasi-balance approximation (i.e., neglecting Dv/Dt) is that the approximation is applied to the meridional wind equation rather than to the divergence equation. As a direct implication, equatorial Kelvin waves are retained, whereas gravity waves, including the mixed Rossby-gravity mode, are filtered from the system.

In section 2 we investigate the question of how the meteorological primitive equations [including Phillips' traditional approximation, from Phillips (1966)] are modified by the neglect of meridional acceleration. The budgets for kinetic energy and potential vorticity are considered and found to take consistent forms with appropriate redefinitions of conserved quantities. Section 3 establishes a prognostic primary circulation and a diagnostic/elliptic secondary circulation. Section 4 treats the issue of invertibility of the potential vorticity to obtain the quasi-balanced mass and wind fields of the primary circulation. In section 5 we discuss the normal modes of the so-called first internal mode of the quasi-balance system, both with a resting basic state and a more representative distribution of tropical easterlies and midlatitude westerlies. Section 6 emphasizes our conclusions and implications for future research.

2. Balance model equations

We develop our quasi-balance system from the hydrostatic primitive equations on a sphere, consisting of a zonal momentum equation, meridional momentum equation, hydrostatic approximation, conservation of mass, and thermodynamic energy equation:

$$\frac{DM}{Dt} = -\frac{\partial\Phi}{\partial\lambda} + F \quad (1a)$$

$$\delta \frac{Dv}{Dt} + \frac{u^2}{a} \tan\phi + (2\Omega \sin\phi)u = -\frac{\partial\Phi}{a\partial\phi} \quad (1b)$$

$$\frac{\partial\Phi}{\partial z} = G(z)\theta \quad (1c)$$

$$\frac{\partial(\rho u)}{a \cos\phi \partial\lambda} + \frac{\partial(\rho v \cos\phi)}{a \cos\phi \partial\phi} + \frac{\partial(\rho w)}{\partial z} = 0 \quad (1d)$$

$$\frac{D\theta}{Dt} = Q. \quad (1e)$$

Here $M \equiv a \cos\phi(\Omega a \cos\phi + u)$ is the angular momentum per unit mass, Φ is geopotential, F is a zonal body torque, a is the earth's radius, λ is longitude, and ϕ is latitude. The vertical coordinate z can represent several different vertical coordinate systems, as indicated in Table 1. The particular coordinate system is arbitrary; the key feature is that the "density function"

$\rho(z)$ and the hydrostatic proportionality coefficient $G(z)$ be prescribed functions only of the vertical coordinate. With the hydrostatic approximation, sound waves are eliminated and a simple one-dimensional relationship is maintained between the mass variable Φ and the temperature variable θ , which we will generally consider to be the potential temperature. In the absence of diabatic heating Q , the variable θ is conserved. The Lagrangian time rate of change following a parcel is D/Dt :

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla = \frac{\partial}{\partial t} + \frac{u}{a \cos\phi} \frac{\partial}{\partial\lambda} + \frac{v}{a} \frac{\partial}{\partial\phi} + w \frac{\partial}{\partial z},$$

where (u, v, w) are the velocity components in the (eastward, northward, upward) directions.

The fundamental balance approximation is in the meridional momentum equation. For clarity, we have introduced a balance tracer δ that equals one in the primitive equations, and which we set to zero in our balance system. "Balance" in this context means a gradient wind type balance between the zonal wind and the meridional pressure gradient force. This dynamical approximation is the logical extension of Gill's (1980) equatorial β -plane, long-wave approximation to the sphere.

a. Scale analysis

The derivation of our balance system from the primitive equations consists of ignoring the meridional acceleration Dv/Dt in (1b) (i.e., $\delta = 0$). In order to examine the conditions under which this approximation is justified, we define the following characteristic scales:

ν^{-1}	time scale
U^*	magnitude of u
V^*	magnitude of v
L_λ^*	spatial scale in λ
L_ϕ^*	spatial scale in ϕ .

With this scaling we estimate the order of magnitude of the important terms in the horizontal momentum equations as

$$\frac{Du}{Dt} - fv \approx -\frac{\partial\Phi}{a \cos\phi \partial\lambda} \quad (2a)$$

$$\nu U^* \quad fV^* \quad \frac{\Delta\Phi}{L_\lambda^*}$$

$$\frac{Dv}{Dt} + fu \approx -\frac{\partial\Phi}{a \partial\phi} \quad (2b)$$

$$\nu V^* \quad fU^* \quad \frac{\Delta\Phi}{L_\phi^*}$$

Following the long-wave approximation, we treat those phenomena for which the zonal scale of variation (L_λ^*) exceeds the meridional scale (L_ϕ^*); i.e., $L_\lambda^* \ll L_\phi^*$. We ask how the meridional acceleration

TABLE 1. Vertical coordinate systems, where p_o and θ_o represent the surface reference states, $\rho_o = p_o/(R\theta_o)$ and the Exner function, π , is defined as $\Pi = c_p(p/p_o)^\kappa$. The parameter $H = c_p\theta_o/g$ is the scale height of an adiabatic atmosphere with potential temperature θ_o . Primed variables represent perturbations from a horizontally averaged state.

Name	z	$\rho(z)$	$G(z)$	Φ
Pressure	$H\left(1 - \frac{p}{p_o}\right)$	ρ_o	$\frac{g}{\theta_o} \kappa \left(1 - \frac{z}{H}\right)^{\kappa-1}$	geopotential
Log pressure	$H \ln\left(\frac{p_o}{p}\right)$	$\rho_o e^{-(z/H)}$	$\frac{g}{\theta_o} \kappa e^{-(\kappa z/H)}$	geopotential
Pseudo-height	$H\left(1 - \left(\frac{p}{p_o}\right)^\kappa\right)$	$\rho_o \left(1 - \frac{z}{H}\right)^{(1-\kappa)/\kappa}$	$\frac{g}{\theta_o}$	geopotential
Height (anelastic)	z	$\rho(z)$	$\frac{g}{\theta_o}$	Π'
Height (Boussinesq)	z	ρ_o	$\frac{g}{\theta_o}$	$\frac{p'}{\rho}$

($Dv/Dt \sim \nu V^*$) compares to the meridional pressure gradient ($\partial\Phi/a\partial\phi \sim \Delta\Phi/L_\phi^*$) for two different cases of the zonal momentum equation (2a):

- Case (i): The zonal acceleration and zonal pressure gradient dominate (as in Kelvin waves), so that

$$\nu U^* \sim \frac{\Delta\Phi}{L_\lambda^*} \quad \text{and} \quad \nu U^* \geq fV^*.$$

Then

$$\left| \frac{Dv/Dt}{\partial\Phi/a\partial\phi} \right| \sim \frac{V^* L_\phi^*}{U^* L_\lambda^*} \leq \frac{\nu L_\phi^*}{f L_\lambda^*}. \quad (2c)$$

- Case (ii): The Coriolis term and zonal pressure gradient dominate (as in long Rossby waves), so that

$$fV^* \sim \frac{\Delta\Phi}{L_\lambda^*} \quad \text{and} \quad \nu U^* \leq fV^*.$$

Then

$$\left| \frac{Dv/Dt}{\partial\Phi/a\partial\phi} \right| \sim \frac{\nu L_\phi^*}{f L_\lambda^*}. \quad (2d)$$

In both cases the meridional acceleration can be neglected in comparison with the meridional pressure gradient so long as the time scale is no faster than the rotational time scale ($\nu/f \leq 1$). For tropical phenomena the appropriate value of f is at a latitude corresponding to one Rossby radius away from the equator, typically around 10° , so that the rotational time scale $f^{-1} \sim 1/2$ day.

We note that meridional advection is retained in our approximated system. This is analogous to the hydrostatic approximation in which the vertical acceleration Dw/Dt is neglected, but vertical advection is retained in the system. For the hydrostatic assumption to be valid, conditions like (2c) and (2d) are invoked

except that the meridional length and velocity scales L_ϕ^* and V^* are replaced by the vertical length and velocity scales L_z^* and W^* . Finally, we note that our approach differs fundamentally from the work of Chang (1977); in his system $v = 0$ so that no meridional advection was allowed.

Lorenz (1960) was among the first to realize that approximations based upon a scaling analysis do not generally maintain analogues of the exact conservation laws. In other words, the combination of scale analysis plus proper conservation laws is a better guarantee of an acceptable approximation than scale analysis alone. With proper conservation relations, the governing dynamical equations can preserve the underlying Hamiltonian structure and retain analogues of all the exact conservation laws (e.g., Salmon 1988; Magnusdottir and Schubert 1990). Thus, it is important for a dynamic prognostic scheme to be characterized by appropriate conservation relations. Derivations of conservation relations with the tracer δ are presented below.

b. Conservation relations

Hoskins (1975) identified four "pseudo-conservation relations" of the primitive equations that should be satisfied by any approximate dynamical system: conservation of potential temperature, conservation of potential vorticity, a three-dimensional vorticity equation, and an energy equation. Since the potential temperature budget is unmodified, there is no need to discuss it explicitly. In fact, our quasi-balance approximation is closely related to the geostrophic momentum approximation of Hoskins: if one takes his full equations with the geostrophic momentum approximation, sets $v_g = 0$ through the long-wave approximation, and allows the full variation of f along with appropriate spherical geometry, one obtains essentially our quasi-balance set.

Equation (1a) states explicitly that angular momentum is conserved in the absence of zonal pressure gradients and body torques. Here we show that kinetic energy and potential vorticity are conserved quantities if they are redefined. Such redefinition follows the procedures used in presently accepted systems, such as the hydrostatic, quasi-geostrophic, and geostrophic momentum approximations.

From (1a) and (1b) we obtain a kinetic energy budget,

$$\frac{D}{Dt} \left(\frac{u^2}{2} + \delta \frac{v^2}{2} \right) = -\mathbf{v} \cdot \nabla \Phi + uF, \quad (3)$$

where \mathbf{v} is the horizontal velocity vector. It is clear from (3) that this balance system (with $\delta = 0$) applies to phenomena in which the zonal velocity predominates over the meridional velocity component.

Equations for the three components of the absolute vorticity vector are obtained by taking the curl of the horizontal momentum equations and the hydrostatic relation. Defining the absolute vorticity vector

$$\boldsymbol{\zeta} \equiv (\xi, \eta, \zeta) = \left(-\delta \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z}, 2\Omega \sin\phi + \delta \frac{\partial v}{a \cos\phi \partial \lambda} - \frac{\partial(u \cos\phi)}{a \cos\phi \partial \phi} \right) \quad (4)$$

and the three-dimensional velocity divergence

$$\nabla \cdot \mathbf{u} \equiv \frac{\partial u}{a \cos\phi \partial \lambda} + \frac{\partial(v \cos\phi)}{a \cos\phi \partial \phi} + \frac{\partial w}{\partial z}, \quad (5)$$

we find

$$\frac{D\xi}{Dt} = (\boldsymbol{\zeta} \cdot \nabla)u - (\nabla \cdot \mathbf{u})\xi + \frac{G}{a} \frac{\partial \theta}{\partial \phi} + \frac{\tan\phi}{a} \frac{\partial}{\partial z} \left(\frac{u^2}{2} + \delta \frac{v^2}{2} \right) \quad (6a)$$

$$\frac{D\eta}{Dt} = (\boldsymbol{\zeta} \cdot \nabla)v - (\nabla \cdot \mathbf{u})\eta - \frac{G}{a \cos\phi} \frac{\partial \theta}{\partial \lambda} + \frac{1}{a \cos\phi} \frac{\partial F}{\partial z} \quad (6b)$$

$$\frac{D\zeta}{Dt} = (\boldsymbol{\zeta} \cdot \nabla)w - (\nabla \cdot \mathbf{u})\zeta - \frac{1}{a \cos\phi} \frac{\partial F}{a \partial \phi}. \quad (6c)$$

We see that the hydrostatic approximation has already eliminated the vertical velocity from the vorticity vector (i.e., $\partial w/\partial y$ in ξ and $-(\partial w/\partial x)$ in η). The zonal balance approximation has the corresponding effect of eliminating the meridional velocity v . As a direct result the zonal component of the vorticity vector vanishes, so that the appropriate vorticity vector lies in the meridional plane and depends only on the zonal component of the flow.

A conservation law for potential vorticity follows from the vorticity relations combined with conservation of mass (1d) and thermodynamic energy (1e).

Defining potential vorticity

$$\begin{aligned} \rho q &\equiv \boldsymbol{\zeta} \cdot \nabla \theta \\ &= -\delta \frac{\partial v}{\partial z} \frac{\partial \theta}{a \cos\phi \partial \lambda} + \frac{\partial u}{\partial z} \frac{\partial \theta}{a \partial \phi} \\ &\quad + \left(2\Omega \sin\phi + \delta \frac{\partial v}{a \cos\phi \partial \lambda} - \frac{\partial u \cos\phi}{a \cos\phi \partial \phi} \right) \frac{\partial \theta}{\partial z} \end{aligned} \quad (7)$$

some manipulation leads to

$$\rho \frac{Dq}{Dt} = \boldsymbol{\zeta} \cdot \nabla Q + \frac{i \cdot (\nabla \theta \times \nabla F)}{a \cos\phi}. \quad (8)$$

Unlike the symmetric case treated by Stevens (1983), the meridional momentum equation is explicitly required in deriving the potential vorticity budget. However, when the balance approximation (1b with $\delta = 0$) is made, the potential vorticity is formally independent of v and $\partial \theta/\partial \lambda$. Only the zonal flow u , and meridional and vertical gradients of potential temperature are involved in the potential vorticity.

3. The primary and secondary circulations

On a sphere it is useful to introduce the latitudinal coordinate $\mu \equiv \sin\phi$ in which equal increments in μ contain equal increments of surface area. Simultaneously, we introduce $U \equiv u \cos\phi$ and $V \equiv v \cos\phi$, which is proportional to the mass flux across a latitude circle. With these definitions, three-dimensional advection becomes

$$\begin{aligned} \mathbf{u} \cdot \nabla &= \frac{u}{a \cos\phi} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \phi} + w \frac{\partial}{\partial z} \\ &= \frac{U}{a(1-\mu^2)} \frac{\partial}{\partial \lambda} + \frac{V}{a} \frac{\partial}{\partial \mu} + w \frac{\partial}{\partial z}. \end{aligned} \quad (9)$$

The continuity equation (1d) can be expressed

$$\begin{aligned} 0 = \nabla \cdot (\rho \mathbf{u}) &= \frac{1}{a(1-\mu^2)} \frac{\partial(\rho U)}{\partial \lambda} \\ &\quad + \frac{1}{a} \frac{\partial(\rho V)}{\partial \mu} + \frac{\partial(\rho w)}{\partial z}. \end{aligned} \quad (1d')$$

Henceforth we set $\delta = 0$ for the zonal wind balance approximation. The gradient wind balance equation (1b) then gives a diagnostic relation between meridional pressure gradient and angular momentum:

$$-f_1(\mu)(M^2 - M_E^2) = \frac{\partial \Phi}{\partial \mu}, \quad (1b')$$

where angular momentum M consists of the earth's angular momentum $M_E \equiv \Omega a^2(1 - \mu^2)$ plus the relative component $M_R \equiv aU$. Here $f_1(\mu)$ is a latitudinal function,

$$f_1(\mu) \equiv \frac{\mu/a^2}{(1 - \mu^2)^2} = \frac{\sin\phi}{a^2 \cos^4\phi}. \quad (10)$$

Thermal wind balance is obtained by cross differentiating (1b') and (1c) to define the baroclinicity B :

$$\rho B \equiv -G \frac{\partial \theta}{\partial \mu} = f_1 \frac{\partial M^2}{\partial z}. \quad (11)$$

The *primary circulation* consists of those dynamical fields involved directly in the balance relations (Φ , θ , M) and derivable directly from them (U , ζ , q). In a prediction scheme, only one of the primary field variables can be consistently prognosed: the others must be diagnosed through the balance relations from the prognostic variable. Since three prognostic relations have appeared—(1a) for angular momentum, (1e) for potential temperature, and (8) for potential vorticity, any one of these three could be used in the time integration procedure. However, for reasons indicated in section 4 on invertibility, it seems that a prediction scheme is best formulated for θ or M rather than the derived quantity q . This aspect differs markedly from other balance systems, such as quasi-geostrophic and semigeostrophic dynamics, in which the time evolution of q is generally calculated.

The *secondary circulation* is the unbalanced component of the flow that is obtained by using thermal wind balance to eliminate $\partial M/\partial t$ and $\partial \theta/\partial t$ between Eqs. (1a) and (1e). It is analogous to the ageostrophic flow calculated from the diagnostic, elliptic omega equation in the quasi-geostrophic system, and consists of the vertical motion and an associated divergent horizontal wind. Two diagnostic elliptic equations arise for the secondary circulation because the flow in the meridional plane is both rotational and divergent. The divergent component can be obtained from the continuity equation, and the rotational component can then be calculated from the differentiated thermal wind balance [cf. Eq. (17) below].

A simpler procedure involving a single elliptic equation is possible if we define an additional variable for the primary circulation. Let \tilde{V} be that part of the meridional flow V that would combine with U for non-divergent flow:

$$\frac{1}{a(1-\mu^2)} \frac{\partial U}{\partial \lambda} + \frac{\partial \tilde{V}}{a \partial \mu} = 0 \quad (12)$$

or

$$\tilde{V} \equiv - \int_{-1}^{\mu} \frac{\partial U}{\partial \lambda} \frac{d\mu}{1-\mu^2} = -a \int_{-1}^{\mu} \frac{\partial \lambda}{\partial \lambda} d\mu. \quad (13)$$

Here $\lambda \equiv U/[a(1-\mu^2)]$ is the relative angular velocity about the polar axis. (Note that \tilde{V} vanishes identically for zonally symmetric flow, but is nonzero for Kelvin waves.)

Then we define $V' \equiv V - \tilde{V}$ as the meridional component of the secondary circulation. With definition (13), the continuity equation (1d) can be expressed without further approximation as

$$\frac{\partial(\rho V')}{a \partial \mu} + \frac{\partial(\rho w)}{\partial z} = 0. \quad (14)$$

Hence we can define a streamfunction ψ for the secondary circulation by

$$\rho V' = - \frac{\partial \psi}{\partial z}; \quad \rho w = \frac{\partial \psi}{a \partial \mu}. \quad (15)$$

It is also worth noting that the vertical component of vorticity and the horizontal divergence can both be expressed as μ -derivatives, consistent with the "long-wave approximation" in which meridional scales are negligible compared to zonal scales:

$$\mathbf{k} \cdot \boldsymbol{\zeta} = - \frac{\partial M}{a^2 \partial \mu} = f - \frac{\partial U}{a \partial \mu}; \quad \nabla \cdot \mathbf{v} = \frac{\partial V'}{a \partial \mu}. \quad (16)$$

When the zonal velocity U is greater than the secondary meridional velocity V' , it follows that the vertical component of relative vorticity will generally exceed the horizontal divergence.

Once we have defined the streamfunction ψ for the secondary circulation, a diagnostic secondary circulation equation immediately follows. Differentiating the thermal wind relation (11) with time

$$-G \frac{\partial}{\partial \mu} \left(\frac{\partial \theta}{\partial t} \right) = f_1 \frac{\partial}{\partial z} \left(2M \frac{\partial M}{\partial t} \right), \quad (17)$$

then substituting from the prognostic equations (1a) and (1e), and finally rearranging terms, we obtain

$$\begin{aligned} \frac{\partial}{\partial \mu} \left(A \frac{\partial \psi}{\partial \mu} + B \frac{\partial \psi}{\partial z} \right) + \frac{\partial}{\partial z} \left(B \frac{\partial \psi}{\partial \mu} + C \frac{\partial \psi}{\partial z} \right) \\ = a \left[\frac{\partial}{\partial \mu} (G\tilde{Q}) + \frac{\partial}{\partial z} (2f_1 M\tilde{F}) \right]. \quad (18) \end{aligned}$$

Here $\rho A \equiv G(\partial \theta/\partial z)$ is proportional to the square of the Brunt-Väisälä frequency (N^2) and represents the gravitational stability. Baroclinicity is represented by ρB , and $\rho C \equiv -f_1(\partial M^2/\partial \mu)$ is proportional to the square of the inertial frequency ($f\zeta$) and represents the inertial stability. The variables \tilde{Q} and \tilde{F} represent the sources of potential temperature and angular momentum, respectively, as modified by the asymmetric pressure gradient force and advections:

$$\tilde{Q} \equiv Q - \left(\lambda \frac{\partial \theta}{\partial \lambda} + \frac{\tilde{V}}{a} \frac{\partial \theta}{\partial \mu} \right) \quad (19a)$$

$$\tilde{F} \equiv F - \left(\lambda \frac{\partial M}{\partial \lambda} + \frac{\tilde{V}}{a} \frac{\partial M}{\partial \mu} \right) - \frac{\partial \Phi}{\partial \lambda}. \quad (19b)$$

Equation (18) is the diagnostic equation for the secondary circulation, just as in the Eliassen (1951) balanced vortex. However, it applies also to the asymmetric balanced flow. Solving (18) is the basis for computing the secondary circulation once the primary circulation, torque F , and heating rate Q are known. A major advantage of our system is that the transient

aspects of geostrophic and gradient adjustment are filtered from the diagnosed circulation.

Since the coefficients in (18) generally depend on longitude as well as latitude and height, this two-dimensional equation must be solved at each longitude or for each zonal Fourier wavenumber. The computational expense of solving (18) repeatedly can be minimized, however, by employing multigrid techniques as demonstrated by Ciesielski et al. (1986). For example his results showed that a multigrid scheme solved an elliptic equation similar to (18) to the level of truncation error 26 times faster than an optimal successive-over-relaxation (SOR) scheme.

The differential equation (18) is elliptic, parabolic or hyperbolic according to whether $AC - B^2$ is greater than, equal to, or less than zero, respectively. In general, (18) is elliptic and is described as such throughout this paper. However in limited regions (e.g., areas of strong anticyclonic flow near the equator), the coefficient C may be negative, resulting in inertial instability. The longwave approximation eliminates the exponentially growing mode of this instability, but does not eliminate the hyperbolic region. To circumvent this problem Dunkerton (1989) has suggested resetting C to zero if it becomes negative. In solving a meridional circulation equation similar to (18), Hack et al. (1989) found that a slightly nonelliptic region near the equator posed no problems when solving their equation with a standard relaxation procedure.

4. The invertibility principle

Hoskins et al. (1985) emphasized the utility of "potential vorticity thinking" in understanding large-scale dynamical processes. A key aspect of this approach is the availability of an invertibility principle for balanced dynamical systems, by which it is possible to obtain the primary circulation field variables uniquely from the spatial distribution of potential vorticity at any instant in time, given appropriate boundary conditions.

Our quasi-balanced system retains this general property with one very significant exception. It is clear from Eqs. (1) that equatorial Kelvin waves are included, since the neglected meridional acceleration vanishes identically for Kelvin waves on the equatorial β -plane. It is also easily shown with linear shallow water equations that Kelvin waves have zero potential vorticity, in common with the general class of Poincaré or inertia-gravity waves. Kelvin waves are invisible on a potential vorticity map. Therefore if one inverts the potential vorticity to obtain the primary circulation, through Eq. (7), we would expect the Kelvin wave component of the primary circulation to be missing.

It is for this reason that we have identified the present approximation as quasi-balanced: a significant part of the time-dependent flow can be predicted but is not related to the potential vorticity field. Thus it appears preferable to predict either the angular momentum or

the potential temperature as the single prognosed variable, rather than the potential vorticity, in a forecast scheme. The temperature variable was predicted in a symmetric calculation on an f -plane by Stevens (1984).

5. Normal modes

In order to test the limits of validity of the quasi-balance system, it is useful to compare the wave characteristics with and without the approximation. For this purpose we have assumed a zonal mean flow that depends only on latitude in a shallow water model on a spherical earth. For a given zonal wavelength, the following governing equations are solved using the method described by Stevens and Ciesielski (1986) to determine the eigenfrequencies and eigenfunctions. We prescribe a global mean equivalent depth $h_o = 250$ m in the linearized shallow water equations:

$$\left(\frac{\partial}{\partial t} + \frac{\bar{u}}{a \cos \phi} \frac{\partial}{\partial \lambda}\right) u' - \left(f - \frac{\partial \bar{u} \cos \phi}{a \cos \phi \partial \phi}\right) v' + \frac{\partial \Phi'}{a \cos \phi \partial \lambda} = 0 \quad (20a)$$

$$\delta \left(\frac{\partial}{\partial t} + \frac{\bar{u}}{a \cos \phi} \frac{\partial}{\partial \lambda}\right) v' + \left(f + \frac{2\bar{u}}{a} \tan \phi\right) u' + \frac{\partial \Phi'}{a \partial \phi} = 0 \quad (20b)$$

$$\left(\frac{\partial}{\partial t} + \frac{\bar{u}}{a \cos \phi} \frac{\partial}{\partial \lambda}\right) \Phi' + \frac{\partial(gh)}{a \partial \phi} v' + gh \left(\frac{\partial u'}{a \cos \phi \partial \lambda} + \frac{\partial v' \cos \phi}{a \cos \phi \partial \phi}\right) = 0. \quad (20c)$$

In these equations $h = h_o + \bar{h}$ represents the unperturbed depth of the fluid, where the variable part of equivalent depth (\bar{h}) was computed with the geostrophic wind relationship in the meridional direction, $\partial(g\bar{h})/a \partial \phi = f\bar{u}$. This form of the shallow water equations is similar to that used by Bennett and Young (1981), except they replaced h by h_o where it multiplies divergence in (20c). This approximation eliminated the troublesome regions of negative equivalent depth (see below) in their problem, but also caused the loss of conservation principles for potential vorticity and energy.

We will consider three sensitivities for predominantly tropical internal waves:

- 1) Equatorial β -plane versus spherical geometry.
- 2) The quasi-balance approximation ($\delta = 0, 1$).
- 3) A basic state at rest versus a climatological flow with tropical easterlies and midlatitude westerlies.

Gill (1980) showed that with no mean flow on an equatorial β -plane, long Rossby and Kelvin wave are nearly nondispersive, and thus can be described as the "long-wave approximation" when $\delta = 0$ is assumed. The fast inertia-gravity waves and the $n = 0$ mixed

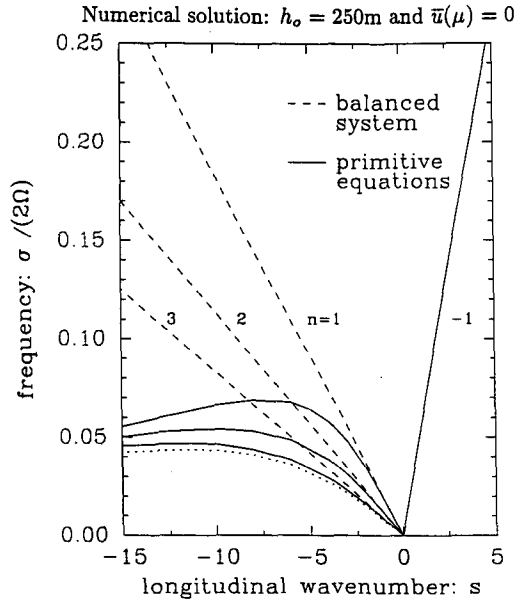


FIG. 1a. Dispersion diagram for the eigenfrequencies computed numerically with a shallow water model on a sphere with a resting basic state. Frequency eigenvalues normalized by (2Ω) are plotted as a function of longitudinal wavenumber (s). Solid lines represent frequencies without approximation ($\delta = 1$); dashed lines for quasi-balanced system ($\delta = 0$) described in the text. Dotted line represents eigenfrequencies for $n = 3$ Rossby mode obtained analytically from equatorial β -plane dispersion relationship. The numerical indices associated with the dashed curves correspond to different meridional structures.

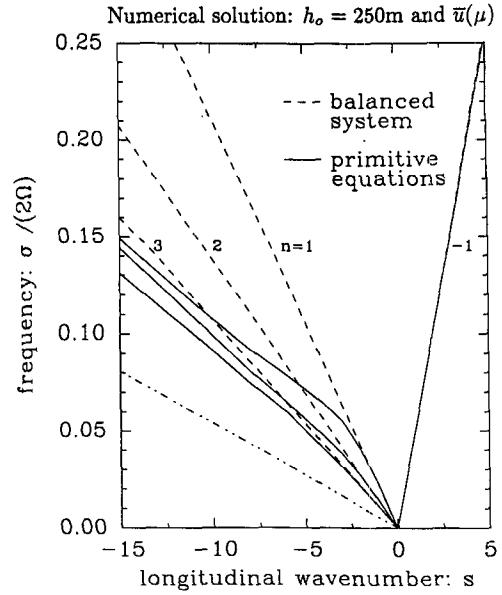


FIG. 1b. Same as Fig. 1a, except computations made using a basic state given by the zonal winds in Fig. 2a. Dashed-dotted line shows the frequencies of a hypothetical disturbance advected westward in a zonal current of -5 m s^{-1} .

Rossby-gravity wave are excluded. The highly dispersive, short-wave (large s) Rossby waves are poorly represented, as might be expected since for these waves meridional velocities typically exceed zonal velocities.

Figure 1 displays the frequency eigenvalues as a function of zonal wavenumber in the low-frequency region of the dispersion diagram. Figure 1a shows the eigenfrequencies on a sphere, with a resting basic state, obtained numerically with the model of Stevens and Ciesielski (1986). In this case where $\bar{u} = 0$, it follows that $\bar{h} = 0$ or $h = h_o$. Figure 1b plots the eigenfrequencies on a sphere in the presence of the zonal mean wind field given in Fig. 2a. Also shown are the absolute vorticity (Fig. 2b), angular momentum (Fig. 2c) and depth of the fluid (Fig. 2d) associated with this wind field. The effects of a nonzero \bar{h} , due primarily to the second term in (20c), are reflected in Fig. 1b by a 1–2% decrease in the eigenfrequencies of the Kelvin mode and up to a 7% increase in the other modes. We also note that the presence of negative fluid depth at higher latitudes (cf. Fig. 2d), the effects of which enter the system through the third term in (20c), results in several nonphysical instabilities that are not relevant to the present discussion.

Under conditions of strongly equatorially trapped waves, the spherical equations (20) asymptotically approximate those on a β -plane for which the following

approximate dispersion relationship for Rossby waves holds:

$$\sigma = \frac{-\beta \left(\frac{s}{a} \right)}{\left(\frac{s}{a} \right)^2 + (2n + 1)\beta(g h_o)^{-1/2}}, \quad (21)$$

where s is the longitudinal wavenumber and $\beta = 2\Omega/a$. According to β -plane theory (Lindzen 1967), the solutions begin their exponential decay at latitudes $\pm\theta_d$. The parameter θ_d is defined as:

$$\theta_d = \epsilon^{-1/4}(2n + 1)^{1/2}, \quad (22)$$

where $\epsilon = (2\Omega a)^2/g h_o$ and n is the number of nodal crossings of the v -eigenfunction. Thus for the case examined here with $h_o = 250 \text{ m}$ ($\epsilon = 352$), the solutions for $n = 1, 2$ and 3 modes begin to decay within 23, 30, and 35 degrees of the equator, respectively. Since the $n = 3$ Rossby mode has the largest latitudinal extent of the ones shown here, β -plane geometry provides the poorest approximation for this mode, and as a consequence, the largest disparity between spherical and β -plane results. This can be seen in Fig. 1a where the dispersion curve from (21) is plotted for the $n = 3$ Rossby mode. The small difference between the two dispersion curves for this mode shows that the β -plane provides an accurate approximation to the sphere for the equivalent depth used here.

With a resting basic state, the Kelvin wave is essentially identical with and without the approximation,

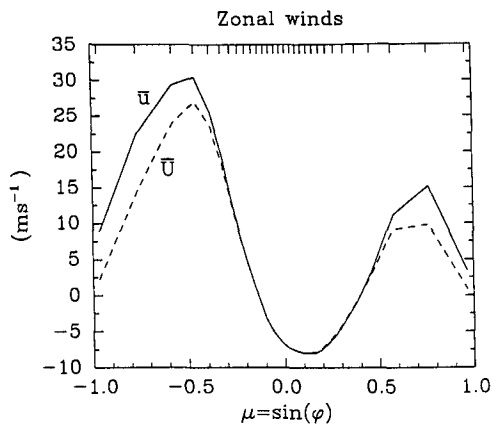


FIG. 2a. Mean (June–August) zonal winds (\bar{u} —from Newell et al. 1974) that were used as basic state for numerical results shown in Fig. 1b plotted as a function of μ (i.e., sine of latitude). Also shown here is $\bar{U} = \bar{u} \cos \phi$, where ϕ is latitude. The location of the model grid points are indicated with tick marks along the top of the figure.

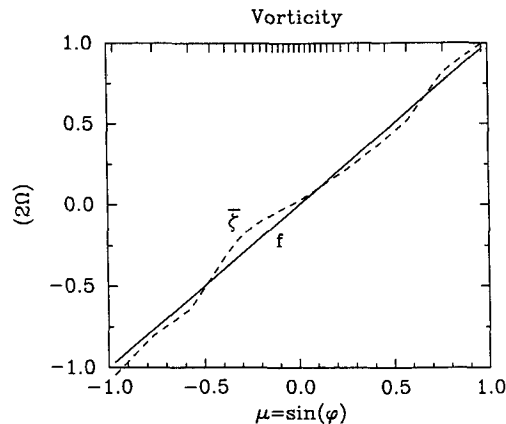


FIG. 2b. Vertical component of earth's vorticity (f) and absolute vorticity (\bar{f}) corresponding to wind profile in Fig. 2a.

whereas Rossby dispersion curves begin to differ noticeably beyond zonal wavenumber 2. In the primitive equations, the group velocity reverses direction at synoptic scales for the gravest modes [$s^2 \approx (2n + 1)\epsilon^{1/2}$].

When a climatological mean flow is assumed, as in Fig. 2a, the nondispersive character of the waves is retained, with the Rossby waves propagating westward somewhat faster than previously. These faster phase speeds are due, in large part, to the Doppler-shifting effect of the easterly flow in which these waves are primarily embedded. The dashed-dotted curve in Fig. 1b shows the frequencies of a hypothetical disturbance advected westward in a zonal current of -5 m s^{-1} . The most noticeable distinction between Figs. 1a and 1b is that in the latter case the Rossby waves in the primitive

equations have much higher frequencies that are significantly closer to the quasi-balanced Rossby waves. In fact, the wavenumber of zero group velocity occurs for wavenumbers ≥ 15 , or zonal wavelengths much shorter than 2700 km. For all the Rossby waves shown, both phase speed and group velocity are westward. The shorter the meridional scale (i.e., the larger n) is the better the approximation. We infer that with representative basic state winds, the quasi-balance approximation might be useful for studying phenomena with zonal scales beyond the lowest few wavenumbers (i.e., $|s| > 2$).

In Fig. 3 eigenfunctions are displayed over one wavelength ($2\pi/s$) for the $n = 1$ Rossby mode at $s = -5$ from Fig. 1. Viewing these eigenfunctions collectively, we note that these modes are confined to within approximately 30 degrees of the equator (i.e., $\mu = \pm 0.5$) which one would expect with $h_o = 250 \text{ m}$. With a resting basic state, the modes are symmetric about the equator with little difference in amplitude between the case with (Fig. 3a) and without (Fig. 3b)

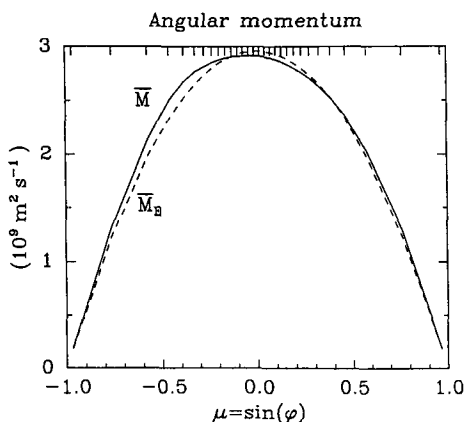


FIG. 2c. Angular momentum per unit mass of earth (\bar{M}_E) and total (\bar{M}) due to earth and zonal winds in Fig. 2a.

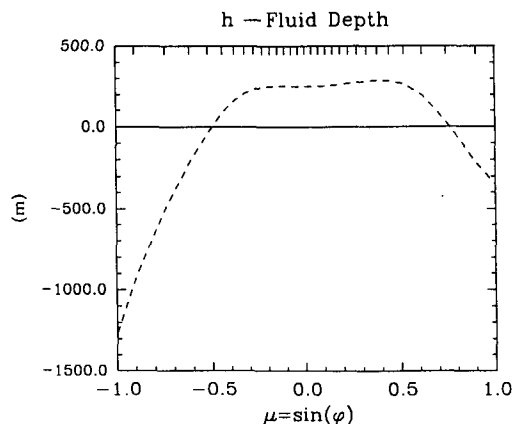


FIG. 2d. Fluid depth (h), where $h = h_o + \bar{h}$, $h_o = 250 \text{ m}$, and \bar{h} is computed using the geostrophic relation and the zonal winds in Fig. 2a.

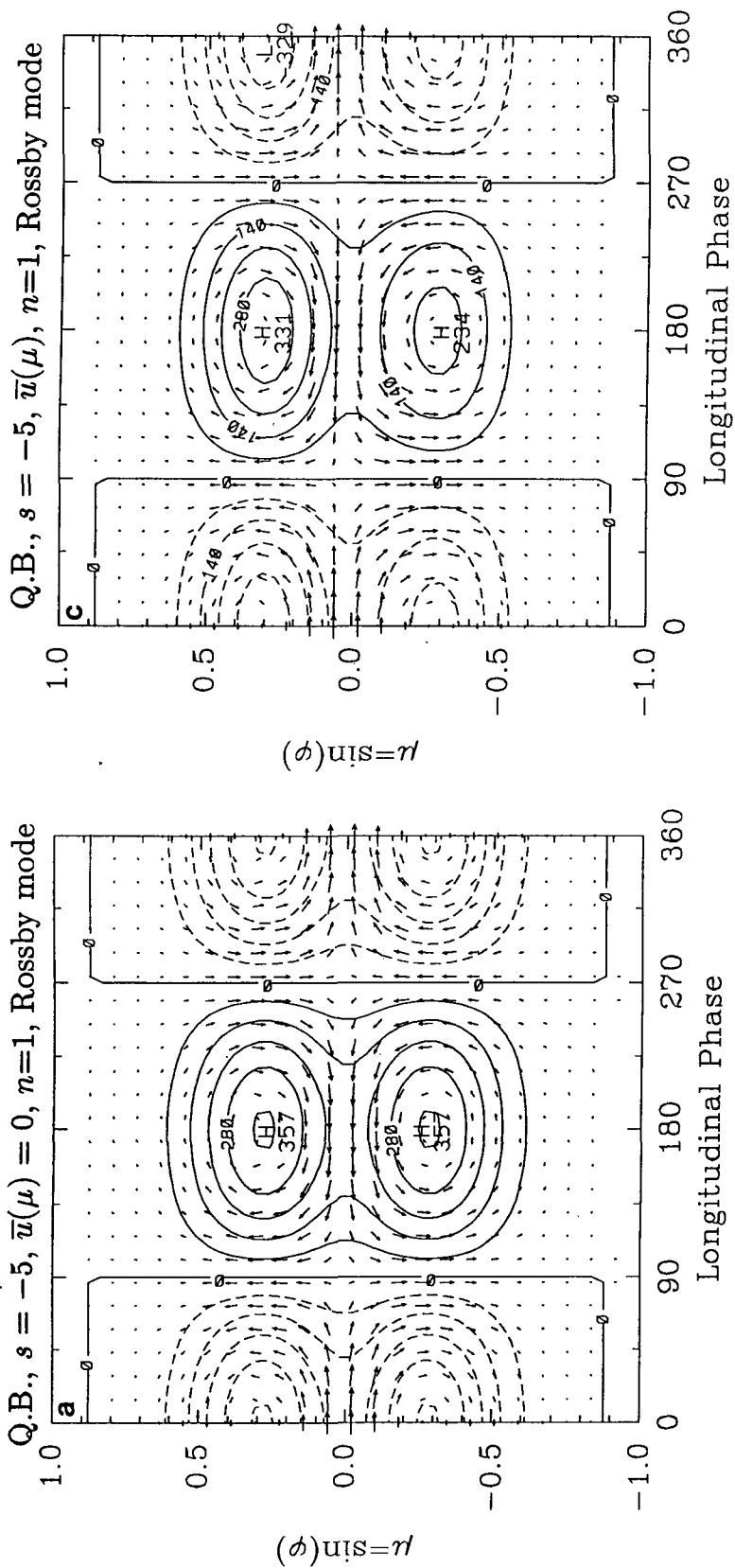


FIG. 3. Nondimensional eigenfunctions for $n = 1$ Rossby mode represented over one wavelength for $s = -5$ and $h_0 = 250$ m. Arrows represent wind velocity normalized by maximum velocity. Solid (dashed) lines show positive (negative) perturbation ψ contours with an interval of 70; values of ψ shown here have been multiplied by 10^4 . Results in various panels

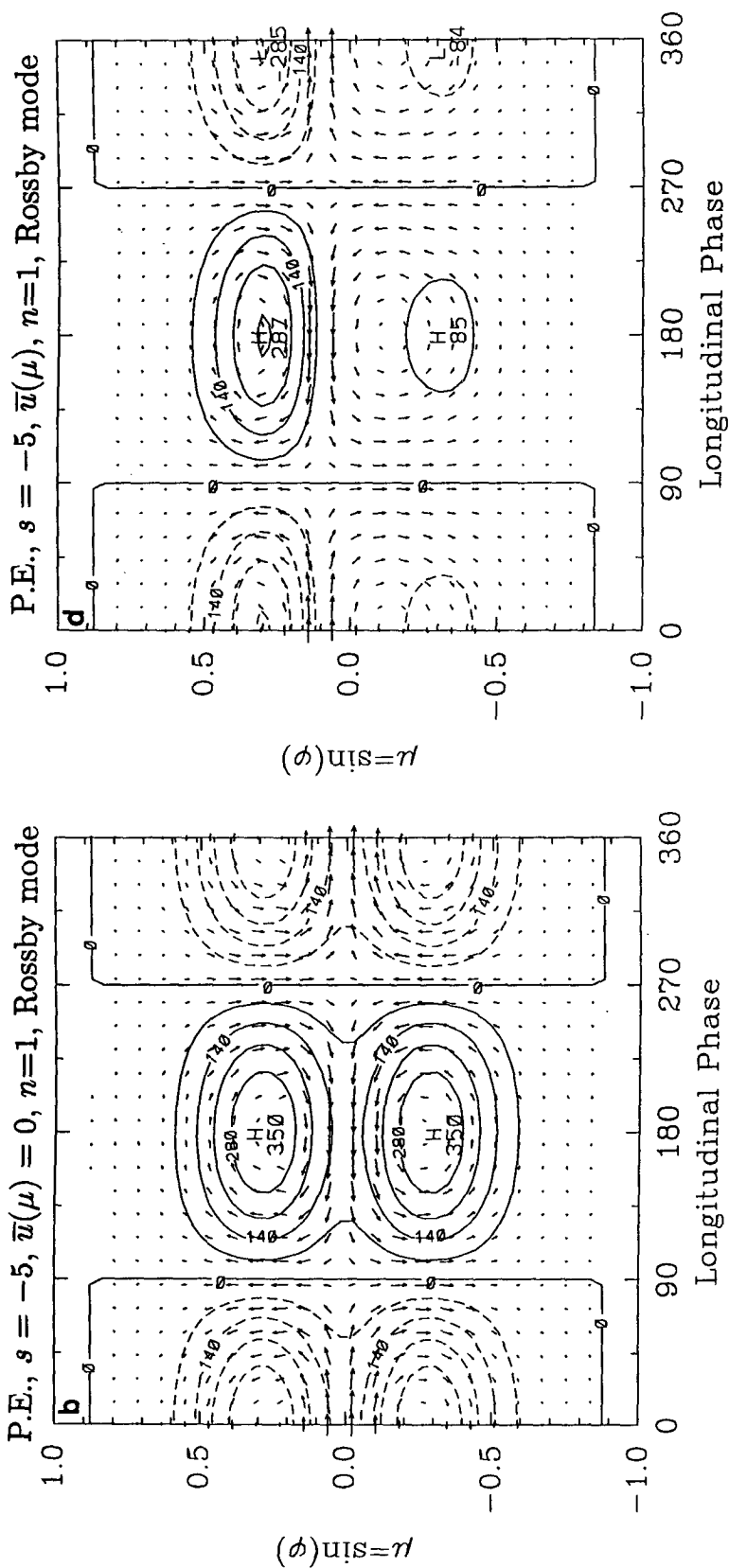


FIG. 3. (Continued) were computed with (a) quasi-balanced system ($\delta = 0$) and basic state at rest, (b) primitive equations ($\delta = 1$) and basic state at rest, (c) quasi-balanced system ($\delta = 0$) and zonal winds in Figs. 2a, (d) primitive equations ($\delta = 1$) and zonal winds in Fig. 2a.

the quasi-balance approximation. In addition, the strong correspondence in the location of the perturbation highs and lows in the Φ' field and the circulation centers, implies a large geostrophic component to the flow. In the presence of the climatological mean flow of Fig. 2a, the modes are no longer purely symmetric about the equator. This is especially noticeable in the $\delta = 1$ case (Fig. 3d) where the response in the Φ' field is a factor of 3 smaller in the southern (winter) hemisphere. In addition, the obvious lack of geostrophic balance in this case suggests that the neglected term, $(\bar{u}/a \cos \phi)(\partial v'/\partial \lambda)$, in (20b) contributes significantly to the primitive-equations modal structure.

6. Discussion and conclusions

A primary reason why midlatitude motions are understood better than tropical motions is that a hierarchy of models exists for midlatitude flows. Since baroclinic instability can be studied with the quasi-geostrophic equations and frontogenesis with semi-geostrophic theory, it is not always necessary to resort to the primitive equations. Unfortunately the present state of modeling tropical motions is not as advanced. For example, what balanced theory can be used to study the 40–50 day oscillation? Quasi-geostrophic theory cannot be used in the tropics, while the geostrophic momentum approximation associated with the semigeostrophic system has, until recently, only been applied to midlatitude f and β planes. As in midlatitudes, our ability to forecast and understand weather in the tropics would be greatly enhanced by a spectrum of dynamical models with varying degrees of complexity. Moreover, a globally valid three-dimensional balanced theory would unify midlatitude and tropical balanced theories into one framework. The balanced system presented in this paper is an effort in this direction, and, in this sense, is complementary to the work of Shutts (1989), who has proposed a planetary semi-geostrophic theory.

In this paper we have proposed a quasi-balance dynamical scheme as a means to isolate slowly evolving meteorological phenomena. Fast inertia–gravity waves are excluded, but equatorial Kelvin waves are retained. This dynamic scheme follows directly from the long-wave approximation of Gill (1980), but is extended to spherical geometry and a stratified fluid. Conceptually, it is founded on the neglect of the meridional acceleration of an air parcel in comparison with the meridional components of the pressure gradient force and the Coriolis force.

With this approximation, a “clean” expression for potential vorticity is derived that depends only on meridional and vertical gradients of the primary circulation variables (u , θ). A key distinction from the quasi-geostrophic system is that vertical advection of potential vorticity is retained. Hence there are both two-dimensional aspects of this approximate system (namely, the neglect of the zonal component of the

vorticity vector, consistent with the long-wave approximation) and three-dimensional aspects (namely, vertical advection) which are likely to be important for tropical motions.

A generalized thermal wind relation, involving the vertical gradient of angular momentum and the meridional gradient of potential temperature, is used to obtain a single diagnostic equation for the secondary circulation. In the limit of zonally symmetric motions, this elliptic equation becomes the Eliassen balanced vortex equation for meridional streamfunction.

Since inertia–gravity waves and inertial instability are excluded, we speculate that the key adjustment process in the tropics is inertial adjustment rather than the geostrophic adjustment of middle latitudes. Specifically, balanced flow refers to inertially stable configurations, rather than two-dimensional, quasi-geostrophic flow. This balance applies to large-scale waves for which the zonal group velocity is in the same direction as the zonal phase speed; it does not apply to phenomena characterized by highly dispersive, small-scale Rossby waves.

With the quasi-balance system, potential vorticity thinking can be applied to tropical dynamics. In fact, we suggest that tropical *climate* dynamics might be definable in terms of such a dynamical system.

To follow up on the results presented here, we intend to investigate with historical data the degree of validity of our approximation in the meridional momentum equation. We suspect that a potential temperature–angular momentum coordinate system for symmetric motions might also be useful for asymmetric motions, particularly in relation to wave–mean flow interactions through the Eliassen–Palm fluxes. We hope to exploit the advantages of this system in both the diagnostic analysis of the observed circulation, and in the medium-range and perhaps long-range forecast of the large-scale flow. Examples of relevant phenomena with long time scales include El Niño/Southern Oscillation, the tropical Hadley and Walker circulations, and the standing waves associated with continental-scale forcing that varies with the seasonal cycle. In the same vein, Shutts (1989) has proposed studying climate dynamics with his similar balanced system.

A similar conceptual framework can also be applied to other geometries with straightforward extensions of the techniques described here. For example, we have begun considering such a balance system in the context of the cylindrical geometry of tropical storms. The predominance of low azimuthal wavenumbers in the flow patterns and the long time scales of such storms relative to the time scales for convection and inertia–gravity wave propagation are similar in principle to the assumptions made herein for a spherical annulus.

Acknowledgments. The authors would like to thank Isaac Held for suggesting that we base our approximation on the ratio of horizontal scales. Thanks also

go to Gail Cordova for her help in preparing this manuscript. This work has been supported by NSF grants ATM-8609731 and ATM-8814541. The National Center for Atmospheric Research provided computer resources.

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