

$e^{\frac{1}{2}(\beta/c)y^2}$ and must be discarded. If $K_w = 0$, the second term on the left hand side of (2.9) vanishes, so that K_e has the acceptable behaviour $e^{-\frac{1}{2}(\beta/c)y^2}$. Thus, (2.7) and (2.8) become

$$\phi_K + cu_K = K(x - ct)e^{-\frac{1}{2}(\beta/c)y^2}, \quad (2.10)$$

$$\phi_K - cu_K = 0, \quad (2.11)$$

where the function $K(x)$ is determined from the initial condition.

Now consider the non-Kelvin part of the flow. Equations (2.5) and (2.6) motivate the representations

$$\phi_\varphi + cu_\varphi = \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial y} - \frac{\beta y}{c} \right) \varphi, \quad (2.12)$$

$$v_\varphi = -\frac{1}{c^2} \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \varphi, \quad (2.13)$$

$$\phi_\varphi - cu_\varphi = \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial y} + \frac{\beta y}{c} \right) \varphi. \quad (2.14)$$

The plausibility of (2.12)–(2.14) is easily checked by noting the equality produced when they are substituted into (2.5) and (2.6). When the Kelvin solution (2.10)–(2.11) and the non-Kelvin representations (2.12)–(2.14) are used in (2.4), we obtain

$$u = -\left(\frac{\partial^2}{\partial x \partial y} + \frac{\beta y}{c^2} \frac{\partial}{\partial t} \right) \varphi + \frac{1}{2c} K(x - ct) e^{-\frac{1}{2}(\beta/c)y^2}, \quad (2.15)$$

$$v = \left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \varphi, \quad (2.16)$$

$$\phi = \left(\frac{\partial^2}{\partial t \partial y} + \beta y \frac{\partial}{\partial x} \right) \varphi + \frac{1}{2} K(x - ct) e^{-\frac{1}{2}(\beta/c)y^2}. \quad (2.17)$$

If we substitute (2.15)–(2.17) back into the original shallow water equations, we find that (2.1) and (2.3) are satisfied, and that (2.2) will also be satisfied if φ is a solution of the equation (Ripa 1994)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\beta^2 y^2}{c^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \frac{\partial \varphi}{\partial t} + \beta \frac{\partial \varphi}{\partial x} = 0. \quad (2.18)$$

If (2.18) can be solved for φ , the u , v , ϕ fields can be easily recovered from (2.15)–(2.17) by differentiation of φ . Because of its central role in the following analysis, we shall refer to (2.18) as the “master equation.”

Since the φ -field yields the non-Kelvin part of the flow, the master equation (2.18) describes the highly divergent flow associated with inertia-gravity waves as well as the quasi-nondivergent, potential vorticity dynamics associated with Rossby waves. In this regard, it is interesting to note that the x -derivative of (2.18) yields the potential vorticity equation, i.e.,

$$\frac{\partial q}{\partial t} + \beta v = 0, \quad (2.19)$$

where

$$q = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{\beta y}{c^2} \phi \quad (2.20)$$

is the potential vorticity anomaly. Using (2.15)–(2.17), the potential vorticity anomaly can be expressed entirely in terms of φ as

$$q = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\beta^2 y^2}{c^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \frac{\partial \varphi}{\partial x} + \frac{\beta}{c^2} \frac{\partial \varphi}{\partial t}. \quad (2.21)$$

Similarly, the y -derivative of (2.18) yields the divergence equation, i.e.,

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \beta y \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \beta u + \nabla^2 \phi = 0. \quad (2.22)$$

In the next section we introduce a filtering approximation that leads to a master equation that is first order in time rather than third order in time. The filtering approximation has no effect on the Kelvin part of the flow, i.e., it filters inertia-gravity waves without distorting Kelvin waves—an extremely useful property for studying the MJO and ENSO.

3. Filtered model

The longwave approximation of (2.1)–(2.3) is a filtering approximation obtained by neglecting $\partial v / \partial t$ in (2.2). Here we consider a more accurate filtered model obtained by approximating (2.1)–(2.3) by

$$\frac{\partial u}{\partial t} - \beta y v + \frac{\partial \phi}{\partial x} = 0, \quad (3.1)$$

$$\frac{\partial \tilde{v}}{\partial t} + \beta y u + \frac{\partial \phi}{\partial y} = 0, \quad (3.2)$$

$$\frac{\partial \phi}{\partial t} + c^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0, \quad (3.3)$$

where \tilde{v} is the approximation of v defined below. Since (3.1) is identical to (2.1), and (3.3) is identical to (2.3), the argument given between (2.4) and (2.17) remains essentially unchanged, but with the inclusion of a representation for \tilde{v} . Thus, the representations of u , v , \tilde{v} , ϕ are

$$u = -\left(\frac{\partial^2}{\partial x \partial y} + \frac{\beta y}{c^2} \frac{\partial}{\partial t} \right) \varphi + \frac{1}{2c} K(x - ct) e^{-\frac{1}{2}(\beta/c)y^2}, \quad (3.4)$$

$$v = \left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \varphi, \quad (3.5)$$

$$\tilde{v} = \frac{\partial^2 \varphi}{\partial x^2}, \quad (3.6)$$

$$\phi = \left(\frac{\partial^2}{\partial t \partial y} + \beta y \frac{\partial}{\partial x} \right) \varphi + \frac{1}{2} K(x - ct) e^{-\frac{1}{2}(\beta/c)y^2}. \quad (3.7)$$

Note that, although (3.4), (3.5), and (3.7) are formally identical to (2.15)–(2.17), the potential function φ in Section 2 is a solution of the (primitive) master equation (2.18), while the

potential function in the present section is a solution of the (filtered) master equation (3.8), given below.

If we substitute (3.4)–(3.7) into the approximate shallow water equations (3.1)–(3.3), we find that (3.1) and (3.3) are satisfied, and that (3.2) will also be satisfied if φ is a solution of the (filtered) master equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\beta^2 y^2}{c^2} \right) \frac{\partial \varphi}{\partial t} + \beta \frac{\partial \varphi}{\partial x} = 0. \quad (3.8)$$

Note that (3.8) is first order in time, while (2.18) is third order in time. As we shall see, (3.8) filters inertia-gravity modes while retaining an accurate description of Rossby modes.

As in the primitive equation case, it is interesting to note that the x -derivative of (3.8) yields the potential vorticity equation, i.e.,

$$\frac{\partial q}{\partial t} + \beta v = 0, \quad (3.9)$$

where

$$\begin{aligned} q &= \frac{\partial \tilde{v}}{\partial x} - \frac{\partial u}{\partial y} - \frac{\beta y}{c^2} \phi \\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\beta^2 y^2}{c^2} \right) \frac{\partial \varphi}{\partial x} + \frac{\beta}{c^2} \frac{\partial \varphi}{\partial t} \end{aligned} \quad (3.10)$$

is the potential vorticity anomaly, while the y -derivative of (3.8) yields

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial y} \right) - \beta y \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \beta u + \nabla^2 \phi = 0. \quad (3.11)$$

Note that the first line of (3.10) is identical to (2.20), except that v has been replaced by \tilde{v} in the vorticity. In addition, note that (3.11) is identical to (2.22), except that v has been replaced by \tilde{v} in the divergence.

It is also interesting to note that, by subtracting $(1/c^2)(\partial/\partial t)$ of (3.8) from $(\partial/\partial x)$ of (3.10), we obtain

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\beta^2 y^2}{c^2} \right) v = \frac{\partial q}{\partial x}, \quad (3.12)$$

which can be considered to be a PV invertibility principle, i.e., a relation that can be used to compute the meridional wind v from the potential vorticity q . Note that (3.9) and (3.12) form a closed system in v and q . In fact, combining $(\partial/\partial t)$ of (3.12) with $(\partial/\partial x)$ of (3.9), we obtain

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\beta^2 y^2}{c^2} \right) \frac{\partial v}{\partial t} + \beta \frac{\partial v}{\partial x} = 0, \quad (3.13)$$

which is identical to (3.8), except for v rather than φ . In Section 5 we choose to proceed with the solution of (3.8) rather than (3.13), since the (u, v, \tilde{v}, ϕ) -fields are so easily recoverable from φ through the use of (3.4)–(3.7).

4. Solution of the master equation for the PE model

We can solve the partial differential equation (2.18) by transforming it into an ordinary differential equation in time, using a Fourier transform in x and a Hermite transform in y . We first take the Fourier transform of (2.18), defining the transform pair

$$\varphi_m(y, t) = \frac{1}{2\pi a} \int_{-\pi a}^{\pi a} \varphi(x, y, t) e^{-imx/a} dx, \quad (4.1)$$

$$\varphi(x, y, t) = \sum_{m=-\infty}^{\infty} \varphi_m(y, t) e^{imx/a}, \quad (4.2)$$

where the integer m denotes the zonal wavenumber. In this way, (2.18) reduces to

$$\left\{ m^2 - \varepsilon^{1/2} \left(\frac{\partial^2}{\partial \hat{y}^2} - \hat{y}^2 \right) + \frac{\varepsilon}{(2\Omega)^2} \frac{\partial^2}{\partial t^2} \right\} \frac{\partial \varphi_m}{\partial t} = 2\Omega i m \varphi_m, \quad (4.3)$$

where

$$\varepsilon = \frac{4\Omega^2 a^2}{c^2} \quad (4.4)$$

is Lamb's parameter and $\hat{y} = (\beta/c)^{1/2} y = \varepsilon^{1/4} (y/a)$ is the dimensionless northward coordinate.

We now convert (4.3) into an ordinary differential equation by transforming in \hat{y} . We use the transform pair

$$\varphi_{mn}(t) = \int_{-\infty}^{\infty} \varphi_m(\hat{y}, t) \mathcal{H}_n(\hat{y}) d\hat{y}, \quad (4.5)$$

$$\varphi_m(\hat{y}, t) = \sum_{n=0}^{\infty} \varphi_{mn}(t) \mathcal{H}_n(\hat{y}), \quad (4.6)$$

where the Hermite functions $\mathcal{H}_n(\hat{y})$ ($n = 0, 1, 2, \dots$) are related to the Hermite polynomials $H_n(\hat{y})$ ($n = 0, 1, 2, \dots$) by $\mathcal{H}_n(\hat{y}) = (\pi^{1/2} 2^n n!)^{-1/2} H_n(\hat{y}) e^{-\frac{1}{2}\hat{y}^2}$. The Hermite functions $\mathcal{H}_n(\hat{y})$ satisfy the recurrence relation

$$\hat{y} \mathcal{H}_n(\hat{y}) = \left(\frac{n+1}{2} \right)^{1/2} \mathcal{H}_{n+1}(\hat{y}) + \left(\frac{n}{2} \right)^{1/2} \mathcal{H}_{n-1}(\hat{y}), \quad (4.7)$$

and the derivative relation

$$\frac{d\mathcal{H}_n(\hat{y})}{d\hat{y}} = - \left(\frac{n+1}{2} \right)^{1/2} \mathcal{H}_{n+1}(\hat{y}) + \left(\frac{n}{2} \right)^{1/2} \mathcal{H}_{n-1}(\hat{y}). \quad (4.8)$$

The first two Hermite functions are $\mathcal{H}_0(\hat{y}) = \pi^{-1/4} e^{-\frac{1}{2}\hat{y}^2}$ and $\mathcal{H}_1(\hat{y}) = 2^{1/2} \pi^{-1/4} \hat{y} e^{-\frac{1}{2}\hat{y}^2}$, from which all succeeding structure functions can be computed using the recurrence relation (4.7). Plots of $\mathcal{H}_n(\hat{y})$ for $n = 0, 1, 2, 3, 4$ are shown in Fig. 1. Note that (4.5) can be obtained through multiplication of (4.6) by $\mathcal{H}_{n'}(\hat{y})$, followed by integration over \hat{y} and use of the orthonormality relation

$$\int_{-\infty}^{\infty} \mathcal{H}_n(\hat{y}) \mathcal{H}_{n'}(\hat{y}) d\hat{y} = \begin{cases} 1 & n' = n, \\ 0 & n' \neq n. \end{cases} \quad (4.9)$$

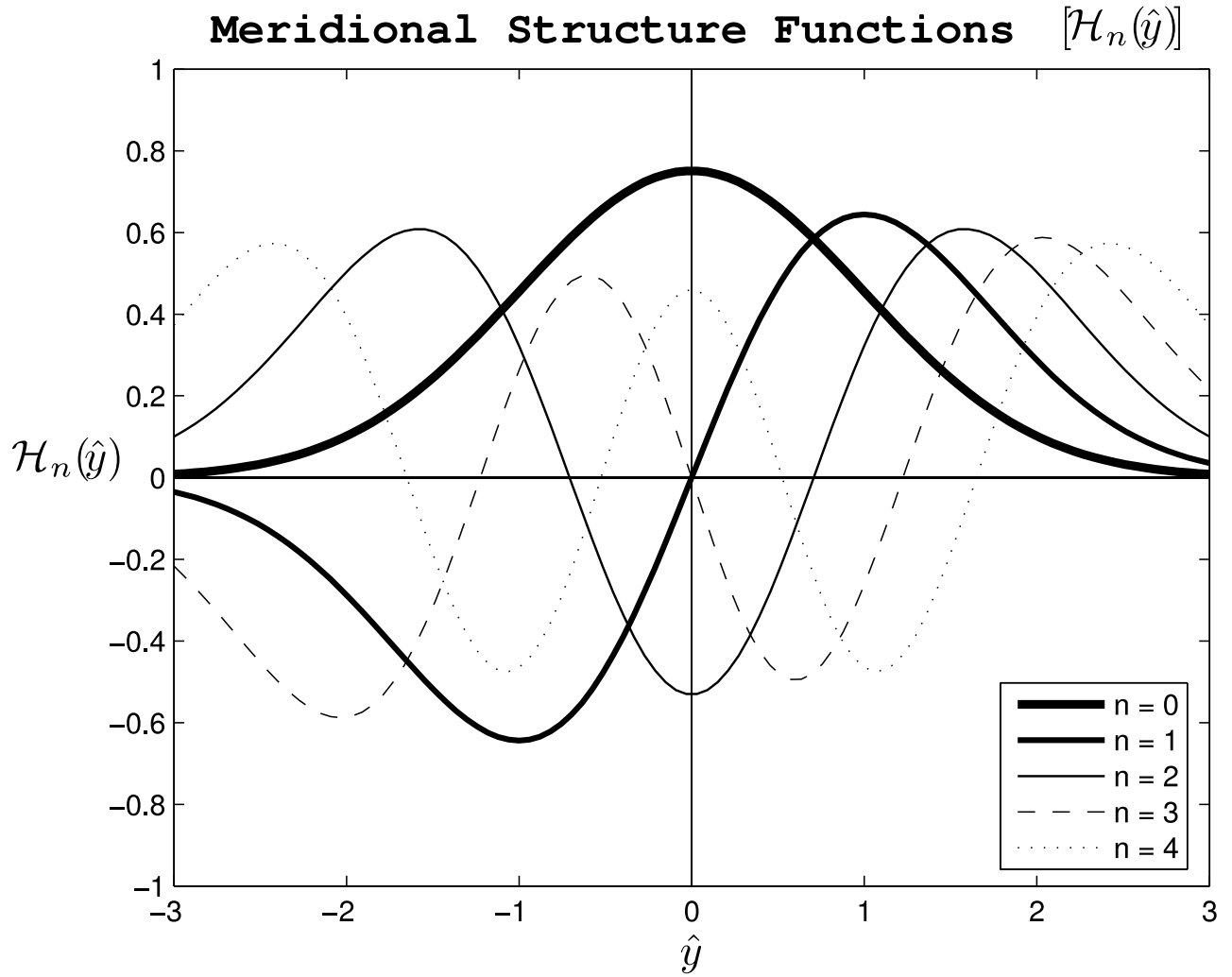


Figure 1. The Hermite functions $\mathcal{H}_n(\hat{y})$ for $n = 0, 1, 2, 3, 4$. These satisfy the orthonormality condition (4.9) and serve as the basis functions for the transform pair (4.5) and (4.6).

Multiplying (4.3) by $\mathcal{H}_n(\hat{y})$ and integrating over \hat{y} (i.e., taking the Hermite transform of (4.3)) we obtain the third order ordinary differential equation

$$\left\{ m^2 + \varepsilon^{1/2}(2n+1) + \frac{\varepsilon}{(2\Omega)^2} \frac{d^2}{dt^2} \right\} \frac{d\varphi_{mn}}{dt} = 2\Omega im\varphi_{mn}. \quad (4.10)$$

In the derivation of (4.10) we have used two integrations by parts (with vanishing boundary terms) and the fact that $\mathcal{H}_n(\hat{y})$ is an eigenfunction of the operator $(d^2/d\hat{y}^2 - \hat{y}^2)$, i.e., $(d^2/d\hat{y}^2 - \hat{y}^2)\mathcal{H}_n(\hat{y}) = -(2n+1)\mathcal{H}_n(\hat{y})$. The solution of (4.10) is

$$\varphi_{mn}(t) = \sum_{r=0}^2 \varphi_{mnr}(0) e^{-iv_{mnr}t}, \quad (4.11)$$

where the dimensionless frequencies $\hat{v}_{mnr} = v_{mnr}/(2\Omega)$ are solutions of

$$\varepsilon \hat{v}_{mnr}^2 - m^2 - \frac{m}{\hat{v}_{mnr}} = \varepsilon^{1/2}(2n+1) \quad (4.12)$$

for $n = 0, 1, 2, \dots$, with $r = 0, 1, 2$ serving as an index for the three roots of the dispersion relation (4.12). Thus, using (4.2), (4.6), and (4.11), we conclude that the solution of the master equation (2.18) is

$$\varphi(x, y, t) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^2 \varphi_{mnr}(0) \mathcal{H}_n(\hat{y}) e^{i(mx/a - v_{mnr}t)}. \quad (4.13)$$

Using the solution (4.13) in the right hand sides of (2.15)–(2.17), and then making use of (4.7) and (4.8), we obtain the final solution

$$\begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^2 \varphi_{mnr}(0) \begin{pmatrix} u_{mnr} \\ v_{mnr} \\ \phi_{mnr} \end{pmatrix} e^{i(mx/a - v_{mnr}t)} + \frac{1}{2} \begin{pmatrix} c^{-1} \\ 0 \\ 1 \end{pmatrix} K(x - ct) e^{-\frac{1}{2}\varepsilon^{1/2}(y/a)^2}, \quad (4.14)$$

where

$$u_{mnr}(y) = \frac{i\varepsilon^{1/4}}{a^2} \left[\left(\frac{n+1}{2} \right)^{1/2} (\varepsilon^{1/2} \hat{v}_{mnr} + m) \mathcal{H}_{n+1}(\hat{y}) + \left(\frac{n}{2} \right)^{1/2} (\varepsilon^{1/2} \hat{v}_{mnr} - m) \mathcal{H}_{n-1}(\hat{y}) \right] \quad (4.15)$$

$$v_{mnr}(y) = \frac{1}{a^2} (\varepsilon \hat{v}_{mnr}^2 - m^2) \mathcal{H}_n(\hat{y}) \quad (4.16)$$

$$\phi_{mnr}(y) = \frac{ic\varepsilon^{1/4}}{a^2} \left[\left(\frac{n+1}{2} \right)^{1/2} (\varepsilon^{1/2} \hat{v}_{mnr} + m) \mathcal{H}_{n+1}(\hat{y}) - \left(\frac{n}{2} \right)^{1/2} (\varepsilon^{1/2} \hat{v}_{mnr} - m) \mathcal{H}_{n-1}(\hat{y}) \right] \quad (4.17)$$

are the eigenfunctions (Matsuno 1966) for the Rossby modes ($r = 0$) and the inertia-gravity modes ($r = 1, 2$).

The dimensionless frequencies \hat{v}_{mnr} , obtained from the primitive equation dispersion relation (4.12) with $\varepsilon = 500$ (which corresponds to $c \approx 41.6 \text{ ms}^{-1}$), are shown by the solid circles in Fig. 2. Note that we have chosen to plot the $n = 0$ modes that are usually discarded in equatorial β -plane theory. The reason is as follows. When $n = 0$, the cubic dispersion relation (4.12) can be factored to yield $(\varepsilon^{1/2} \hat{v} + m)(\varepsilon^{1/2} \hat{v}^2 - m\hat{v} - 1) = 0$. In Matsuno's (1966) original argument, which deals with solutions of an equation for v , the $\varepsilon^{1/2} \hat{v} = -m$ eigenvalues and their associated eigenfunctions for v were justifiably discarded. In the present argument, which deals with solutions of an equation for ϕ , the $\varepsilon^{1/2} \hat{v} = -m$ eigenvalues and their associated eigenfunctions for ϕ can either be discarded or retained in the summation (4.13). To see this, first note that their contribution (using the index $r = 0$) to the summation (4.13) is

$$\begin{aligned} \pi^{-1/4} e^{-\frac{1}{2}(\beta/c)y^2} \sum_{m=-\infty}^{\infty} \varphi_{m,0,0}(0) e^{i(m/a)(x+ct)} \\ \equiv e^{-\frac{1}{2}(\beta/c)y^2} F(x+ct) \equiv G(x+ct, y). \end{aligned} \quad (4.18)$$

Referring to the operators in the large parentheses of (2.15)–(2.17), now note that

$$\left(\frac{\partial^2}{\partial x \partial y} + \frac{\beta y}{c^2} \frac{\partial}{\partial t} \right) G = 0, \quad (4.19)$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G = 0, \quad (4.20)$$

$$\left(\frac{\partial^2}{\partial t \partial y} + \beta y \frac{\partial}{\partial x} \right) G = 0. \quad (4.21)$$

Thus, (4.18) makes no contribution to the u, v, ϕ fields, so that $G(x+ct, y)$ represents a gauge transformation of the ϕ -field that leaves the u, v, ϕ fields unchanged. Another way of seeing this is to simply note that the right hand sides of (4.15)–(4.17)

vanish when $n = 0$ and $\varepsilon^{1/2} \hat{v}_{m,0,0} = -m$. For convenience in understanding the filtering properties discussed in the next section, we have included the $\varepsilon^{1/2} \hat{v}_{m,0,0} = -m$ root in Fig. 2, with the label “ $n = 0$ (gauge).” Finally, for completeness we have also plotted (open circles) the Kelvin wave frequencies, which are given by $\varepsilon^{1/2} \hat{v} = m$. However, it should be remembered that it is only the non-Kelvin part of the flow that is described by the master equation (2.18) and the potential ϕ .

5. Solution of the master equation for the filtered model

The solution of (3.8) proceeds in a manner analogous to the solution of (2.18). After transforming in x and y , we obtain the first order ordinary differential equation

$$\frac{d\varphi_{mn}}{dt} = \frac{2\Omega im}{m^2 + \varepsilon^{1/2}(2n+1)} \varphi_{mn}. \quad (5.1)$$

The solution of (5.1) is

$$\varphi_{mn}(t) = \varphi_{mn}(0) e^{-iv_{mn}t}, \quad (5.2)$$

where the dimensionless frequency $\hat{v}_{mn} = v_{mn}/(2\Omega)$ is given by

$$\hat{v}_{mn} = -\frac{m}{m^2 + \varepsilon^{1/2}(2n+1)} \quad (5.3)$$

for $n = 0, 1, 2, \dots$, which is an approximation of the low frequency solutions of the cubic equation (4.12). Thus, we conclude that the solution of (3.8) is

$$\varphi(x, y, t) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \varphi_{mn}(0) \mathcal{H}_n(\hat{y}) e^{i(mx/a - v_{mn}t)}. \quad (5.4)$$

Using the solution (5.4) in the right hand sides of (3.4), (3.5) and (3.7), we obtain

$$\begin{aligned} \begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \varphi_{mn}(0) \begin{pmatrix} u_{mn}(y) \\ v_{mn}(y) \\ \phi_{mn}(y) \end{pmatrix} e^{i(mx/a - v_{mn}t)} \\ + \frac{1}{2} \begin{pmatrix} c^{-1} \\ 0 \\ 1 \end{pmatrix} K(x-ct) e^{-\frac{1}{2}\varepsilon^{1/2}(y/a)^2}, \end{aligned} \quad (5.5)$$

where

$$u_{mn}(y) = \frac{i\varepsilon^{1/4}}{a^2} \left[\left(\frac{n+1}{2} \right)^{1/2} (\varepsilon^{1/2} \hat{v}_{mn} + m) \mathcal{H}_{n+1}(\hat{y}) + \left(\frac{n}{2} \right)^{1/2} (\varepsilon^{1/2} \hat{v}_{mn} - m) \mathcal{H}_{n-1}(\hat{y}) \right] \quad (5.6)$$

$$v_{mn}(y) = \frac{1}{a^2} (\varepsilon \hat{v}_{mn}^2 - m^2) \mathcal{H}_n(\hat{y}), \quad (5.7)$$

$$\begin{aligned} \phi_{mn}(y) = \frac{ic\varepsilon^{1/4}}{a^2} \left[\left(\frac{n+1}{2} \right)^{1/2} (\varepsilon^{1/2} \hat{v}_{mn} + m) \mathcal{H}_{n+1}(\hat{y}) - \left(\frac{n}{2} \right)^{1/2} (\varepsilon^{1/2} \hat{v}_{mn} - m) \mathcal{H}_{n-1}(\hat{y}) \right] \end{aligned} \quad (5.8)$$

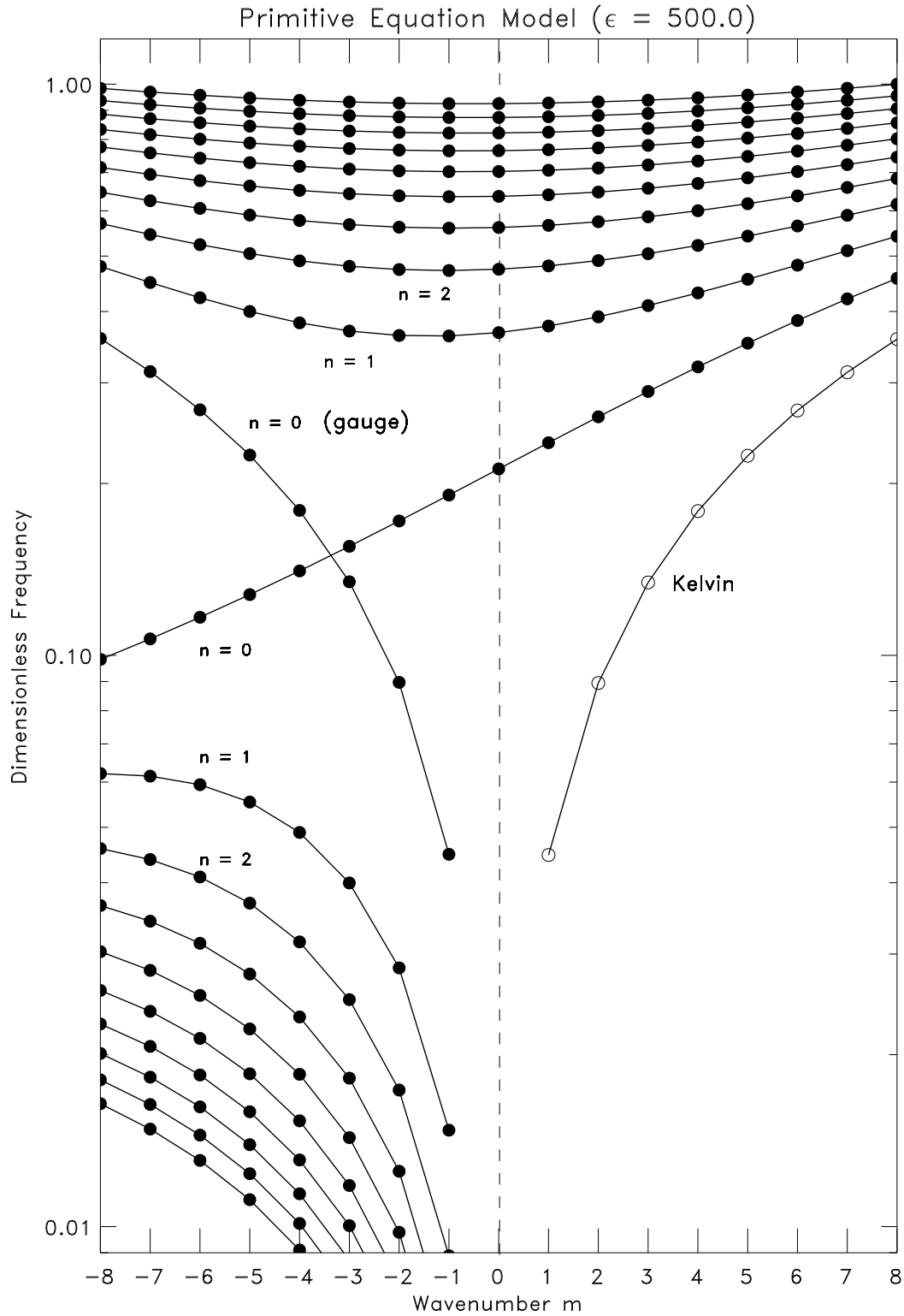


Figure 2. The dimensionless frequencies $\hat{\nu}_{mnr}$ determined from the primitive equation dispersion relation (4.12) with $\epsilon = 500$ and $n = 0, 1, 2, \dots, 9$, are shown with solid circles. Although they make no contribution to the physical fields u, v, ϕ , the $n = 0$ modes labelled "gauge" are included here since they help in the interpretation of Fig. 3. For completeness, the dimensionless Kelvin wave frequencies $\hat{\nu} = \epsilon^{-1/2}m$ are also shown (open circles).

are the eigenfunctions for the Rossby modes. In comparing the balanced model results (5.5)–(5.8) with the primitive equation model results (4.14)–(4.17), we note that the subscripts r and the sum over r are missing in (5.5)–(5.8) since inertia-gravity waves have been filtered. The eigenfunctions in (5.6)–(5.8) are accurate approximations to the Rossby wave eigenfunctions in (4.15)–(4.17) since the only difference is that \hat{v}_{mn} is computed from (5.3) for use in (5.6)–(5.8), while \hat{v}_{mn0} is the low frequency solution of (4.12) for use in (4.15)–(4.17). The filtered dimensionless frequencies (5.3) are plotted in Fig. 3, along with the Kelvin mode $\varepsilon^{1/2}\hat{v} = m$, for the Lamb's parameter $\varepsilon = 500$. A comparison of Figs. 2 and 3 shows how accurately this approximation represents Rossby waves. Note the lack of inertia-gravity waves and that the $n = 0$ modes in Fig. 3 can be interpreted as approximations of the low frequency $n = 0$ Rossby modes for large $|m|$ and approximations of the low frequency $n = 0$ gauge modes for small $|m|$. Since the gauge modes in Fig. 2 make no contribution to u, v, ϕ for the primitive equation model, we may expect that the “approximate gauge modes” in Fig. 3 (i.e., the $n = 0$ modes for $m = -1, -2, -3$) make small contributions to u, v, ϕ in the filtered model.

In summary, we have introduced a new equatorial β -plane filtered model that retains Rossby and Kelvin modes, and that acts as an effective filter of inertia-gravity modes. The new filtered model leads to the Rossby wave dispersion relation (5.3), which is more accurate than the one obtained from the long-wave approximation. In fact, the Rossby wave dispersion relation obtained from the longwave approximation is similar to (5.3), but does not contain the m^2 factor in the denominator of (5.3). Thus, the longwave approximation leads to a catastrophe for high wavenumber Rossby waves.

6. Comparison with a traditional filtering method

The filtering approximation introduced in Section 3 is based on a partitioning of the flow into Kelvin and non-Kelvin parts. More traditional filtering approximations (e.g., Schubert and Masarik 2006) are based on a partitioning of the flow into irrotational and nondivergent parts, i.e., $u = \partial\chi/\partial x - \partial\psi/\partial y$ and $v = \partial\chi/\partial y + \partial\psi/\partial x$, where χ is the velocity potential and ψ is the streamfunction. This partitioning allows us to write the potential vorticity equation (2.19) as

$$\frac{\partial}{\partial t} \left(\nabla^2 \psi - \frac{\beta y}{c^2} \phi \right) + \beta \left(\frac{\partial \chi}{\partial y} + \frac{\partial \psi}{\partial x} \right) = 0, \quad (6.1)$$

and the divergence equation (2.22) as

$$\frac{\partial}{\partial t} \nabla^2 \chi + \nabla \cdot (\nabla \phi - \beta y \nabla \psi) + \beta \frac{\partial \chi}{\partial x} = 0. \quad (6.2)$$

The “local linear balance approximation” of (6.1) and (6.2) is obtained by neglecting the $\partial\chi/\partial y$ term in (6.1), neglecting the first and last terms on the left hand side of (6.2), and treating the βy factor in the middle term of (6.2) as slowly varying.

These approximations lead to $\nabla^2(\phi - \beta y \psi) = 0$, from which it follows that $\phi = \beta y \psi$ and

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\beta^2 y^2}{c^2} \right) \frac{\partial \psi}{\partial t} + \beta \frac{\partial \psi}{\partial x} = 0. \quad (6.3)$$

Note that (6.3) is formally identical to (3.8) and (3.13), but that the dependent variable in (6.3) is the streamfunction ψ , while the dependent variable in (3.8) is the non-Kelvin potential ϕ and the dependent variable in (3.13) is the meridional velocity component v . A result of the identical form of (3.8), (3.13), and (6.3) is that the eigenvalue relation (5.3) results from each. However, the eigenfunctions for the wind and mass fields that are obtained from (6.3) are not as accurate as those obtained from (3.8) and (3.13), as we shall now see.

Using the same Fourier and Hermite transform methods introduced in Sections 4 and 5, we can show that the solution of (6.3) is

$$\psi(x, y, t) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \psi_{mn}(0) \mathcal{H}_n(\hat{y}) e^{i(mx/a - v_{mn}t)}, \quad (6.4)$$

where the nondimensional frequency $\hat{v}_{mn} = v_{mn}/(2\Omega)$ is given by (5.3). Using the solution (6.4) in $u_\psi = -\partial\psi/\partial y$, $v_\psi = \partial\psi/\partial x$, and $\phi = \beta y \psi$, we obtain

$$\begin{pmatrix} u_\psi \\ v_\psi \\ \phi \end{pmatrix} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \psi_{mn}(0) \begin{pmatrix} U_{mn}(y) \\ V_{mn}(y) \\ \Phi_{mn}(y) \end{pmatrix} e^{i(mx/a - v_{mn}t)}, \quad (6.5)$$

where

$$U_{mn}(y) = \frac{\varepsilon^{1/4}}{a} \left[\left(\frac{n+1}{2} \right)^{1/2} \mathcal{H}_{n+1}(\hat{y}) - \left(\frac{n}{2} \right)^{1/2} \mathcal{H}_{n-1}(\hat{y}) \right] \quad (6.6)$$

$$V_{mn}(y) = \frac{im}{a} \mathcal{H}_n(\hat{y}), \quad (6.7)$$

$$\begin{aligned} \Phi_{mn}(y) &= \frac{c\varepsilon^{1/4}}{a} \hat{y} \mathcal{H}_n(\hat{y}) \\ &= \frac{c\varepsilon^{1/4}}{a} \left[\left(\frac{n+1}{2} \right)^{1/2} \mathcal{H}_{n+1}(\hat{y}) + \left(\frac{n}{2} \right)^{1/2} \mathcal{H}_{n-1}(\hat{y}) \right]. \end{aligned} \quad (6.8)$$

Equation (6.5) is an approximation of the top line in (5.5), and hence also an approximation of the Rossby wave part of the top line in (4.14). This is easily confirmed by noting that the top line in (5.5) reduces to (6.5) if $\varepsilon^{1/2}\hat{v}_{mn} + m \rightarrow m$ and $i(m/a)\varphi_{mn}(0) \rightarrow \psi_{mn}(0)$. The approximation (6.5) is not as accurate as the approximation (5.5), a fact that is most apparent in the geopotential field. For example, note that (6.8) yields $\phi = 0$ at $y = 0$ for all n , so that the approximate Rossby wave eigenfunctions (6.8) have no zonal pressure gradient force at the equator. This is not a property of the primitive equation eigenfunctions (4.17) or the approximate eigenfunctions (5.8). A more detailed comparison of such differences

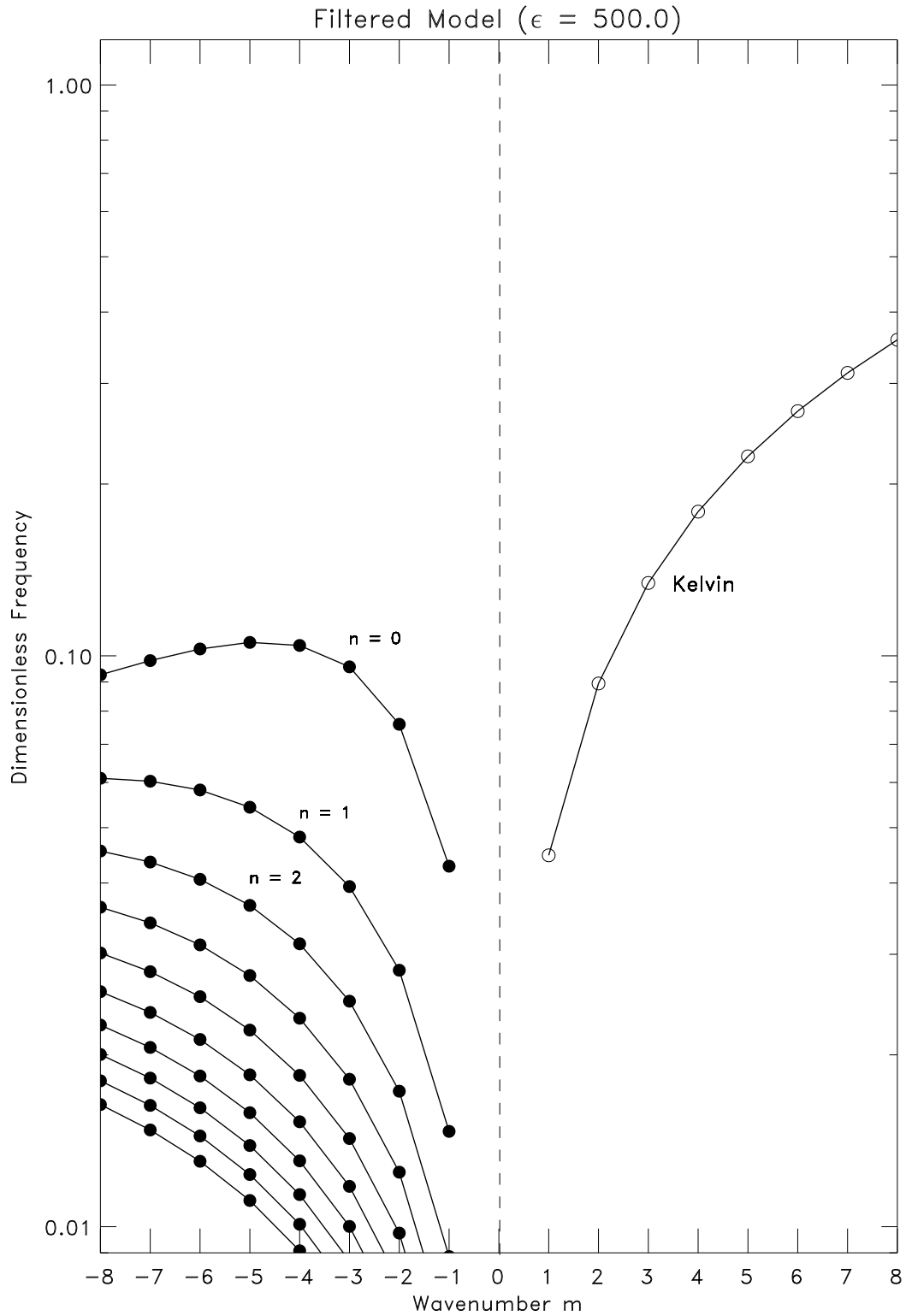


Figure 3. The dimensionless frequencies $\hat{\nu}_{mn}$, determined from the filtered model dispersion relation (5.3) with $\epsilon = 500$ and $n = 0, 1, 2, \dots, 9$, are shown with solid circles. For completeness, the dimensionless Kelvin wave frequencies $\hat{\nu} = \epsilon^{-1/2}m$ are also shown (open circles). In comparing this figure with the primitive equation result shown in Fig. 2, note that all $n \geq 1$ inertia-gravity modes have been filtered, all $n \geq 1$ Rossby modes have been retained, high frequency $n = 0$ gauge modes and inertia-gravity modes have been filtered, while low frequency $n = 0$ gauge modes and Rossby modes have been retained. Also note that the filtering procedure has no effect on the Kelvin modes.

for a forced problem can be found in Schubert and Masarik (2006, their Figs. 3 and 8). The important conclusion to be noted here is that the filtered model of Section 3 is much preferable to the filtered model of this section, not only because of more accurate Rossby wave eigenfunctions but also because of the inclusion of Kelvin waves.

7. Concluding remarks

Filtered models are usually derived by partitioning the flow into nondivergent and irrotational parts, which are expressed in terms of the streamfunction and velocity potential. Then certain approximations are introduced into the divergence and vorticity equations, with the result that inertia-gravity waves are filtered. Such procedures have the disadvantage that, in the process of filtering the inertia-gravity waves, the Kelvin waves are distorted (e.g., Moura 1976). This distortion of the Kelvin waves makes such models of limited use in studying the MJO and ENSO, which require accurate Rossby wave dynamics for the flow on the west side of the heat source and accurate Kelvin wave dynamics for the flow on the east side of the heat source.

In the present paper we have taken a different approach to the filtering problem. We have partitioned the flow into Kelvin and non-Kelvin parts, and expressed the non-Kelvin part in terms of a single potential φ , which satisfies a master equation. We have then approximated the master equation in such a way that the inertia-gravity waves are filtered and the Rossby waves are accurately described. This approach to filtering the inertia-gravity waves leaves the Kelvin waves undistorted and results in a filtered model that is useful for studying a wide range of tropical phenomena, including not only the MJO and ENSO but also smaller scale phenomena in which short Rossby wave energy is significant.

As was discussed in Section 4, the representation of the non-Kelvin part of the flow in terms of the potential φ is not unique, since the gauge transformation $\varphi \rightarrow \varphi + G(x + ct, y)$ leaves the u, v, ϕ fields unchanged (i.e., the u, v, ϕ fields are gauge invariant with respect to $G(x + ct, y)$). This property of φ should not be taken as an argument against its usefulness, since there are many analogous situations in other branches of physics. For example, in Maxwell's theory of electromagnetism the electric and magnetic fields can be written in terms of the scalar potential Φ and the vector potential \mathbf{A} as $\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}$. For a general scalar function Ψ , the gauge transformation $\Phi \rightarrow \Phi + \partial\Psi/\partial t$ and $\mathbf{A} \rightarrow \mathbf{A} - \nabla\Psi$ leaves the \mathbf{E} and \mathbf{B} fields unchanged (i.e., the \mathbf{E} and \mathbf{B} fields are gauge invariant). Thus, in analogy with electromagnetism, it is important to keep in mind that the φ field cannot be measured directly, but can only be inferred from the u, v, ϕ fields to within the gauge field $G(x + ct, y)$.

Can we generalize the previous arguments to include forcing, dissipation, continuous stratification, nonlinearity, and spherical geometry? It appears that all these generalizations are possible, although some subtleties are involved. For ex-

ample, the generalization to the sphere requires performing an analysis similar to that done for the equatorial β -plane in (2.1)–(2.18). On the sphere, this analysis requires an approximation if one is to obtain an analytically tractable master equation. To understand this, consider substituting the representation (2.12) into the left hand side of (2.5) and the representation (2.13) into the right hand side of (2.5). Because of the commutative property of the differential operators, the usefulness of the representations (2.12) and (2.13) is confirmed. On the sphere, this commutative property is lost because of the convergence of the meridians. Thus, the spherical analogue of the analysis (2.1)–(2.18) involves an approximation that depends on the slow variation of $\cos(\text{latitude})$ factors. With this approximation we are led to the following (dimensional) master equation on the sphere:

$$\left(\nabla^2 - \frac{\varepsilon\mu^2}{a^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \frac{\partial\varphi}{\partial t} + \frac{2\Omega}{a^2} \frac{\partial\varphi}{\partial\lambda} = 0, \quad (7.1)$$

where λ is the longitude, μ is the sine of the latitude, and ∇^2 is the horizontal Laplacian operator in spherical coordinates. If (7.1) can be solved for φ , then the u, v, ϕ fields can be recovered by differentiation of φ . As before, the φ -field yields the non-Kelvin part of the flow. Wave solutions of (7.1) are of the form $\varphi(\lambda, \mu, t) = \mathcal{S}_{mn}(\varepsilon; \lambda, \mu) e^{-i\nu_{mn}t}$, where the spheroidal harmonics $\mathcal{S}_{mn}(\varepsilon; \lambda, \mu)$ satisfy $(a^2\nabla^2 - \varepsilon\mu^2)\mathcal{S}_{mn} = -\alpha_{mn}(\varepsilon)\mathcal{S}_{mn}$, with $-\alpha_{mn}(\varepsilon)$ denoting the spheroidal harmonic eigenvalue. When $\varepsilon = 0$, the spheroidal harmonics reduce to the spherical harmonics, with the eigenvalues $\alpha_{mn}(0) = n(n+1)$. When the spheroidal harmonic wave form is substituted into (7.1) we obtain the primitive equation dispersion relation

$$\varepsilon\hat{\nu}_{mn}^2 - \frac{m}{\hat{\nu}_{mn}} = \alpha_{mn}(\varepsilon), \quad (7.2)$$

where $\hat{\nu}_{mn} = \nu_{mn}/(2\Omega)$ is the dimensionless frequency. Equation (7.2) is the spherical generalization of (4.12).

As in the equatorial β -plane case, the filtered version of (7.1) is

$$\left(\nabla^2 - \frac{\varepsilon\mu^2}{a^2} \right) \frac{\partial\varphi}{\partial t} + \frac{2\Omega}{a^2} \frac{\partial\varphi}{\partial\lambda} = 0. \quad (7.3)$$

Searching for solutions of (7.3) having the spheroidal harmonic form, we obtain the Rossby-Haurwitz wave dispersion relation

$$\hat{\nu}_{mn} = -\frac{m}{\alpha_{mn}(\varepsilon)}, \quad (7.4)$$

which is the spherical version of (5.3). The dispersion relation (7.4) is known (see Schubert et al. 2009) to accurately approximate the low frequency modes found numerically by Longuet-Higgins (1968) for the shallow water primitive equations on the sphere. When $\varepsilon = 0$, (7.4) reduces to $\hat{\nu}_{mn} = -m/[n(n+1)]$, which is the well-known result for nondivergent barotropic Rossby-Haurwitz waves on the sphere.

From this brief discussion we conclude that the basic ideas developed in the context of equatorial β -plane theory can in-

deed be generalized to the sphere. A more complete analysis of the spherical case is the topic of current research.

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