Potential Vorticity in a Moist Atmosphere

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ABSTRACT

The potential vorticity principle for a nonhydrostatic, moist, precipitating atmosphere is derived. An appropriate generalization of the well-known (dry) Ertel potential vorticity is found to be $P = \rho^{-1}(2\boldsymbol{\Omega} + \nabla \times \mathbf{u}) \cdot \nabla\theta_\rho$, where ρ is the total density, consisting of the sum of the densities of dry air, airborne moisture (vapor and cloud condensate), and precipitation; $\mathbf{u}$ is the velocity of the dry air and airborne moisture; and $\theta_\rho = T_\rho(p_0/p)^{R_a/c_{Pa}}$ is the virtual potential temperature, with $T_\rho = p/(\rho R_a)$ the virtual temperature, p the total pressure (the sum of the partial pressures of dry air and water vapor), p_0 the constant reference pressure, R_a the gas constant for dry air, and c_{Pa} the specific heat at constant pressure for dry air. Since θ_ρ is a function of total density and total pressure only, its use as the thermodynamic variable in P leads to the annihilation of the solenoidal term, that is, $\nabla\theta_\rho \cdot (\nabla\rho \times \nabla p) = 0$. In the special case of an absolutely dry atmosphere, P reduces to the usual (dry) Ertel potential vorticity.

For balanced flows, there exists an invertibility principle that determines the balanced mass and wind fields from the spatial distribution of P. It is the existence of this invertibility principle that makes P such a fundamentally important dynamical variable. In other words, P (in conjunction with the boundary conditions associated with the invertibility principle) carries all the essential dynamical information about the slowly evolving balanced part of the flow.

1. Introduction

At present, the dynamical basis for global numerical weather prediction and climate models is the set of quasi-static primitive equations. Since the nonhydrostatic motions typical of dry and moist atmospheric convection have such small horizontal scales, and since they cannot be accurately simulated with the quasi-static primitive equations, the collective effects of these motions must be parameterized in quasi-static primitive equation models. However, in the not-too-distant future, it will be possible to construct and run global numerical weather prediction and climate models based on the exact primitive equations, with much more accurate treatments of the moist thermodynamics, and with cloud-resolving spatial discretization. In such nonhydrostatic models, cumulus cloud fields will be explicitly simulated, eliminating the need for cumulus parameterization. This pushes the frontier of empiricism back to the parameterization of the microphysics of the precipitation process.

The purpose of the present paper is to extend the potential vorticity conservation principle to nonhydrostatic models with Ooyama's (1990, 2001) form of moist dynamics and thermodynamics. We begin by reviewing the exact, nonhydrostatic primitive equations for a moist atmosphere in section 2. In section 3 we derive the generalized potential vorticity principle. Equations (20) and (21) are our main results, the latter form being useful for physical interpretation. In section 4 we discuss one of the many possible invertibility principles (depending on the particular balance conditions) associated with the generalized potential vorticity. Section 5 contains a derivation of the ''equivalent potential vorticity'' and a discussion of why this form is unacceptable from the standpoint of possessing an invertibility principle. Conclusions are given in section 6.

2. Nonhydrostatic primitive equations for a moist atmosphere

Consider atmospheric matter to consist of dry air, airborne moisture (vapor and cloud condensate), and pre-

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cipitation. Let ρ_a denote the mass density of dry air,[1] $\rho_m = \rho_v + \rho_c$ the mass density of airborne moisture (consisting of the sum of the mass densities of water vapor ρ_v and airborne condensed water ρ_c), and ρ_r the mass density of precipitating water substance. The total mass density ρ is given by $\rho = \rho_a + \rho_v + \rho_c + \rho_r = \rho_a + \rho_m + \rho_r$. The flux forms of the prognostic equations for ρ, ρ_m, and ρ_r are given in (1)–(3), in which $\mathbf{u}$ denotes the velocity (relative to the rotating earth) of dry air and airborne moisture, and $\mathbf{u} + \mathbf{U}$ denotes the velocity (relative to the rotating earth) of the precipitating water substance, so that $\mathbf{U}$ is the velocity of the precipitating water substance relative to the dry air and airborne moisture. The term Q_r, on the right-hand sides of (2) and (3), is the rate of conversion from airborne moisture to precipitation; this term can be positive (e.g., the collection of cloud droplets by rain) or negative (e.g., the evaporation of precipitation falling through unsaturated air).

The total entropy density is $\sigma = \sigma_a + \sigma_m + \sigma_r$, consisting of the sum of the entropy densities of dry air, airborne moisture, and precipitation. Since the entropy flux is given by $\sigma_a\mathbf{u} + \sigma_m\mathbf{u} + \sigma_r(\mathbf{u} + \mathbf{U}) = \sigma\mathbf{u} + \sigma_r\mathbf{U}$, we can write the flux form of the entropy conservation principle as (4), where Q_σ denotes diabatic processes such as radiation.

Next, we can write the equation of motion as (5), where $\boldsymbol{\zeta} = 2\boldsymbol{\Omega} + \nabla \times \mathbf{u}$ is the absolute vorticity; Φ the potential for the sum of the Newtonian gravitational force and the centrifugal force; $\rho^{-1}\nabla p$ the pressure gradient force, with $p = p_a + p_v$ the sum of the partial pressures of dry air p_a and water vapor p_v; and $\mathbf{F}$ the frictional force per unit mass. The derivation of (5) is given in Ooyama (2001) and is reviewed in appendix B.

Consolidating our results so far, the prognostic equations for the total mass density ρ, the mass density of airborne moisture ρ_m, the mass density of precipitation content ρ_r, the entropy density σ, and the three-dimensional velocity vector $\mathbf{u}$ are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u} + \rho_r\mathbf{U}) = 0, \quad (1)$$

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m\mathbf{u}) = -Q_r, \quad (2)$$

$$\frac{\partial \rho_r}{\partial t} + \nabla \cdot [\rho_r(\mathbf{u} + \mathbf{U})] = Q_r, \quad (3)$$

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma\mathbf{u} + \sigma_r\mathbf{U}) = Q_\sigma, \quad (4)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\zeta} \times \mathbf{u} + \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \Phi\right) + \frac{1}{\rho}\nabla p = \mathbf{F}. \quad (5)$$

The diagnostic variables appearing explicitly in (1)–(5) are the terminal fall velocity $\mathbf{U}$, the entropy density of precipitation σ_r, the source terms Q_r and Q_σ, and the total pressure p. In the determination of $\mathbf{U}$, σ_r, Q_r, Q_σ, and p, several other diagnostic variables are introduced. The additional diagnostic variables introduced in (6)–(13) are the mass density of dry air ρ_a, the thermodynamically possible temperatures T_1 and T_2, the actual temperature T, the partial pressure of dry air p_a, and the partial pressure of water vapor p_v, the formulas for which are given below. To summarize, the diagnostic relations are

$$\rho_a = \rho - \rho_m - \rho_r, \quad (6)$$

$$S_2(\rho_a, \rho_m + \rho_r, T_2) = \sigma, \quad (7)$$

$$\sigma_r = \rho_r C(T_2), \quad (8)$$

$$S_1(\rho_a, \rho_m, T_1) = \sigma - \sigma_r, \quad (9)$$

$$T = \max(T_1, T_2), \quad (10)$$

$$p_a = \rho_a R_a T, \quad (11)$$

$$\begin{cases} \rho_v = \rho_m, \quad \rho_c = 0, \quad p_v = \rho_v R_v T, \\ \quad \text{if } T = T_1 > T_2 \quad \text{(absence of condensate)}, \\ \rho_v = \rho_v^*(T), \quad \rho_c = \rho_m - \rho_v, \quad p_v = E(T), \\ \quad \text{if } T = T_2 > T_1 \quad \text{(saturated vapor)}, \end{cases} \quad (12)$$

$$p = p_a + p_v. \quad (13)$$

Note that, although there are four types of matter (with densities ρ_a, ρ_v, ρ_c, ρ_r), there are only three prognostic mass continuity equations [(1)–(3)]. The separation of the predicted total airborne moisture density ρ_m into the vapor density ρ_v and the cloud condensate density ρ_c is accomplished diagnostically in the two alternatives of (12). In the absence of condensate, all the predicted airborne moisture ρ_m is in vapor form so that $\rho_v = \rho_m$ and $\rho_c = 0$, while if the vapor is saturated, $\rho_v = \rho_v^*(T)$ and $\rho_c = \rho_m - \rho_v$. The formulas for the entropy density functions S_1 and S_2, from which the temperatures T_1 and T_2 are diagnosed, are given in appendix A.

In the context of numerical modeling, the procedure for advancing from one time level to the next consists of computing new values of the prognostic variables ρ, ρ_m, ρ_r, σ, and $\mathbf{u}$ from (1)–(5). The diagnostic variables required for the prognostic stage are determined by sequential evaluation of (6)–(13), namely, the diagnosis of the dry air density ρ_a from (6), the thermodynamically possible (wet bulb) temperature T_2 from (7), the entropy density of precipitation from (8), the thermodynamically possible temperature T_1 from (9), the choice of the actual temperature from (10), the dry air partial pressure p_a from (11), the water vapor mass density ρ_v, the airborne condensate mass density ρ_c, and the water vapor partial pressure p_v from the appropriate condition in (12), and the total pressure from (13). Since they are not essential to our discussion here, we have omitted the parameterization formulas for the terminal fall velocity $\mathbf{U}$ and the source terms Q_r and Q_σ. See Ooyama (2001) for further discussion.

[1] The notation used here follows Ooyama (2001). A list of symbols is given at the end of the paper.

To obtain a better feel for the somewhat unfamiliar system (1)–(13), it is interesting to note the limiting forms of these equations for the case of a perfectly dry atmosphere (i.e., $\rho_v = \rho_c = \rho_r = 0$). In that case the $\rho_r\mathbf{U}$ term in (1) vanishes, Eqs. (2) and (3) are dropped, the $\sigma_r\mathbf{U}$ term in (4) vanishes, and the precipitation contribution to $\mathbf{F}$ (see appendix B) in (5) vanishes. In addition, the diagnostic equations (7), (8), (10), and (12) are dropped; (6) and (13) reduce to $\rho = \rho_a$ and $p = p_a$; (9) reduces to $c_{Va}\ln(T/T_0) - R_a\ln(\rho/\rho_{a0}) = \sigma/\rho$, from which T is diagnosed; and (11) reduces to $p = \rho R_a T$, from which p is computed.

At present it is not feasible to numerically integrate moist, nonhydrostatic, ''full physics'' models over the whole globe with 1–2-km resolution. However, it is possible to perform such 1–2-km-resolution integrations over a single hurricane, for example. Such integrations advance the art of hurricane modeling to a new level that involves much less physical parameterization. In order that such full physics models can be interpreted in terms of well-established principles of geophysical fluid dynamics, we now derive the potential vorticity principle associated with the system (1)–(13). As we shall see, the only equations in the set (1)–(13) that are needed for the derivation of the potential vorticity principle are (1) and (5).

3. The potential vorticity (PV) equation

Consider the fundamental identities $\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$ and $\nabla\cdot(A\mathbf{a}) = A\nabla\cdot\mathbf{a} + \mathbf{a}\cdot\nabla A$, for the arbitrary vector fields $\mathbf{a}$ and $\mathbf{b}$, and the arbitrary scalar field A. Choosing $\mathbf{a} = \nabla\psi$ and $\mathbf{b} = \partial\mathbf{u}/\partial t$ in the first identity, we obtain $\nabla\cdot[\nabla\psi\times(\partial\mathbf{u}/\partial t)] = -\nabla\psi\cdot(\partial\boldsymbol{\zeta}/\partial t)$, since $\nabla\times\nabla\psi = 0$ and $\boldsymbol{\zeta} = 2\boldsymbol{\Omega} + \nabla\times\mathbf{u}$. Choosing $A = \partial\psi/\partial t$ and $\mathbf{a} = \boldsymbol{\zeta}$ in the second identity, we obtain $\nabla\cdot[\boldsymbol{\zeta}(\partial\psi/\partial t)] = \boldsymbol{\zeta}\cdot\nabla(\partial\psi/\partial t)$, since $\nabla\cdot\boldsymbol{\zeta} = 0$. Taking the difference of these two results we obtain

$$\frac{\partial}{\partial t}(\boldsymbol{\zeta}\cdot\nabla\psi) + \nabla\cdot\left(\nabla\psi\times\frac{\partial\mathbf{u}}{\partial t} - \boldsymbol{\zeta}\frac{\partial\psi}{\partial t}\right) = 0. \quad (14)$$

Defining $\dot{\psi} \equiv D\psi/Dt$, where $D/Dt = \partial/\partial t + \mathbf{u}\cdot\nabla$ is the material derivative, and using the triple vector product $\boldsymbol{\zeta}(\mathbf{u}\cdot\nabla\psi) = \mathbf{u}(\boldsymbol{\zeta}\cdot\nabla\psi) + \nabla\psi\times(\boldsymbol{\zeta}\times\mathbf{u})$, we can write (14) as

$$\frac{\partial}{\partial t}(\boldsymbol{\zeta}\cdot\nabla\psi) + \nabla\cdot\left[\mathbf{u}(\boldsymbol{\zeta}\cdot\nabla\psi) + \nabla\psi\times\left(\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\zeta}\times\mathbf{u}\right)\right]$$
$$= \boldsymbol{\zeta}\cdot\nabla\dot{\psi}. \quad (15)$$

Using the continuity equation (1), in the form $D\rho/Dt + \rho\nabla\cdot\mathbf{u} + \nabla\cdot(\rho_r\mathbf{U}) = 0$, we can put (15) in advective form and eliminate $\nabla\cdot\mathbf{u}$ to obtain

$$\rho\frac{D}{Dt}\left(\frac{1}{\rho}\boldsymbol{\zeta}\cdot\nabla\psi\right) + \nabla\cdot\left[\nabla\psi\times\left(\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\zeta}\times\mathbf{u}\right)\right]$$
$$= \boldsymbol{\zeta}\cdot\nabla\dot{\psi} + \left(\frac{1}{\rho}\boldsymbol{\zeta}\cdot\nabla\psi\right)\nabla\cdot(\rho_r\mathbf{U}). \quad (16)$$

We now combine (16) with the momentum equation (5). Substituting from (5) for $\partial\mathbf{u}/\partial t + \boldsymbol{\zeta}\times\mathbf{u}$, noting that $\nabla\cdot(\nabla A\times\nabla B) = 0$ for any scalar functions A and B, (16) reduces to

$$\rho\frac{D}{Dt}\left(\frac{1}{\rho}\boldsymbol{\zeta}\cdot\nabla\psi\right) = \frac{1}{\rho^2}\nabla\psi\cdot(\nabla\rho\times\nabla p) + (\nabla\times\mathbf{F})\cdot\nabla\psi$$
$$+ \boldsymbol{\zeta}\cdot\nabla\dot{\psi} + \left(\frac{1}{\rho}\boldsymbol{\zeta}\cdot\nabla\psi\right)\nabla\cdot(\rho_r\mathbf{U}). \quad (17)$$

For two-dimensional flows with line symmetry or circular symmetry, the vectors $\nabla\psi$, $\nabla\rho$, and ∇p all lie in the same plane so that $\nabla\rho\times\nabla p$ is perpendicular to $\nabla\psi$, and the solenoidal term $\nabla\psi\cdot(\nabla\rho\times\nabla p)$ vanishes no matter how ψ is chosen. However, for general three-dimensional flows, how do we choose the scalar function ψ in such a way that the solenoidal term $\nabla\psi\cdot(\nabla\rho\times\nabla p)$ vanishes? To accomplish this, we follow Ooyama (1990) and first define the virtual temperature T_ρ by

$$T_\rho = \frac{p}{\rho R_a} = \frac{p_a + p_v}{(\rho_a + \rho_m + \rho_r)R_a}, \quad (18)$$

so that T_ρ is the temperature that dry air would have if its pressure and density were equal to those of the given sample of moist air. Note that the virtual temperature T_ρ is equal to the actual temperature T when $p_v = 0$ and $\rho_m = \rho_r = 0$. Next let us define the virtual potential temperature θ_ρ by

$$\theta_\rho = T_\rho\left(\frac{p_0}{p}\right)^\kappa = \frac{p}{\rho R_a}\left(\frac{p_0}{p}\right)^\kappa, \quad (19)$$

where p_0 is a constant reference pressure and $\kappa = R_a/c_{Pa}$. Since θ_ρ can be written as a function of p and ρ only, we have $\nabla\theta_\rho = (\partial\theta_\rho/\partial p)_\rho\nabla p + (\partial\theta_\rho/\partial\rho)_p\nabla\rho$. Since $\nabla p\cdot(\nabla\rho\times\nabla p) = 0$ and $\nabla\rho\cdot(\nabla\rho\times\nabla p) = 0$, this implies that $\nabla\theta_\rho\cdot(\nabla\rho\times\nabla p) = 0$, so the choice[2] $\psi = \theta_\rho$ annihilates the first term on the right-hand side of (17). Thus, (17) reduces to

$$\frac{DP}{Dt} = \frac{1}{\rho}[(\nabla\times\mathbf{F})\cdot\nabla\theta_\rho + \boldsymbol{\zeta}\cdot\nabla\dot{\theta}_\rho + P\nabla\cdot(\rho_r\mathbf{U})], \quad (20a)$$

where

$$P = \frac{1}{\rho}\boldsymbol{\zeta}\cdot\nabla\theta_\rho. \quad (20b)$$

[2] Other choices of ψ involving functions of θ_ρ [e.g., the ''virtual entropy'' $\psi = c_{Pa}\ln(\theta_\rho/T_0)$] could also be used. For consistency with the most widely used definition of potential vorticity in the dry case, we shall confine our attention to the choice $\psi = \theta_\rho$.

The generalized potential vorticity principle (20) is our main result. It should be noted that (20) is a generalization of the well-known (dry) Ertel (1942) potential vorticity principle in three respects: (i) the total density ρ consists of the sum of the densities of dry air, airborne moisture, and precipitation; (ii) θ_ρ is the chosen scalar field; and (iii) precipitation effects are included in $\mathbf{F}$ and in the last term of (20a). In a completely dry atmosphere, the total density ρ reduces to the dry air density, the virtual potential temperature θ_ρ reduces to the ordinary dry potential temperature, and precipitation effects disappear, so that (20) then reduces to the ordinary (dry) Ertel PV principle.

It is worth noting that (20a) can be written in a form that is more physically revealing for thermally forced, frictionally controlled, balanced atmospheric motions. Defining $\mathbf{j} = \boldsymbol{\nabla}\theta_\rho/|\boldsymbol{\nabla}\theta_\rho|$ as the unit vector normal to the θ_ρ surface and $\mathbf{k} = \boldsymbol{\zeta}/|\boldsymbol{\zeta}|$ as the unit vector pointing along the absolute vorticity vector becomes (20a)

$$\frac{DP}{Dt} = P\left(\frac{\mathbf{j}\cdot\boldsymbol{\nabla}\times\mathbf{F}}{\mathbf{j}\cdot\boldsymbol{\zeta}} + \frac{\mathbf{k}\cdot\boldsymbol{\nabla}\dot{\theta}_\rho}{\mathbf{k}\cdot\boldsymbol{\nabla}\theta_\rho} + \frac{\boldsymbol{\nabla}\cdot(\rho_r\mathbf{U})}{\rho}\right). \quad (21)$$

This form emphasizes the exponential nature of the time behavior of P for material parcels moving through a region with approximately constant rate processes associated with $\mathbf{F}$, $\dot{\theta}_\rho$, and $\rho_r\mathbf{U}$. For example, in the intense convective region of a hurricane, $\mathbf{k}$ tends to point upward and radially outward. Since $\dot{\theta}_\rho$ tends to be a maximum at midtropospheric levels, air parcels flowing inward at low levels and spiraling upward in the convective eyewall experience a material increase in P due to the $P(\mathbf{k}\cdot\boldsymbol{\nabla}\dot{\theta}_\rho)/(\mathbf{k}\cdot\boldsymbol{\nabla}\theta_\rho)$ term. This material increase of P can be especially rapid in lower tropospheric regions near the eyewall, where both P and $(\mathbf{k}\cdot\boldsymbol{\nabla}\dot{\theta}_\rho)/(\mathbf{k}\cdot\boldsymbol{\nabla}\theta_\rho)$ are large. Although the $(\mathbf{k}\cdot\boldsymbol{\nabla}\dot{\theta}_\rho)/(\mathbf{k}\cdot\boldsymbol{\nabla}\theta_\rho)$ term reverses sign in the upper troposphere, large values of P are often found there because the large lower tropospheric values are carried upward into the upper troposphere.

It is also worth noting that the effects of precipitation contained in the last term of (20a) can be distributed over the other three terms to obtain an equation that is formally similar to the one usually given for the dry case. This alternative form is discussed in appendix C.

4. Invertibility principle

It is natural to ask if, under certain balance conditions, there exists an invertibility principle for P. The importance of the existence of an invertibility principle is hard to overemphasize. It is the existence of such a principle that makes P such a dynamically interesting quantity. To see that such a principle exists, consider an f-plane case in which a large-scale axisymmetric flow has a slowly evolving $P(r, z, t)$ field. The slow evolution of the P field is due to radial and vertical advection of P, and the $\dot{\theta}_\rho$, $\mathbf{F}$, and $\rho_r\mathbf{U}$ terms on the right-hand side of (20a). Since the evolution is slow, we can consider the tangential wind and mass fields as continuously changing from one hydrostatic and gradient balanced state to another. If we have a method of predicting $P(r, z, t)$, the invertibility problem is to diagnostically determine, at each time, the tangential wind $v(r, z, t)$, the total density $\rho(r, z, t)$, the total pressure $p(r, z, t)$, the virtual temperature $T_\rho(r, z, t)$, and the virtual potential temperature $\theta_\rho(r, z, t)$ from

$$\rho\left(f + \frac{v}{r}\right)v = \frac{\partial p}{\partial r}, \quad (22)$$

$$-g\rho = \frac{\partial p}{\partial z}, \quad (23)$$

$$T_\rho = \frac{p}{\rho R_a}, \quad (24)$$

$$\theta_\rho = T_\rho\left(\frac{p_0}{p}\right)^\kappa, \quad (25)$$

$$\frac{1}{\rho}\left\{-\frac{\partial v}{\partial z}\frac{\partial \theta_\rho}{\partial r} + \left[f + \frac{\partial(rv)}{r\partial r}\right]\frac{\partial \theta_\rho}{\partial z}\right\} = P. \quad (26)$$

This constitutes a system of five equations for the five unknowns $v(r, z, t)$, $\rho(r, z, t)$, $p(r, z, t)$, $T_\rho(r, z, t)$, and $\theta_\rho(r, z, t)$ with given $P(r, z, t)$. Note that the solution of the invertibility problem gives us the total density $\rho(r, z, t)$, the total pressure $p(r, z, t)$, and the virtual temperature $T_\rho(r, z, t)$. The determination of the actual temperature T, the partition of p between p_a and p_v, and the partition of ρ between ρ_a, ρ_m, and ρ_r is not possible from knowledge of P only. In other words, solution of the invertibility problem gives only the parts of the mass field that are of direct dynamical significance.

In principle it is possible to use (22)–(25) to eliminate v, ρ, and θ_ρ from (26) and thereby obtain a single partial differential equation relating the total pressure $p(r, z, t)$ to the known PV distribution $P(r, z, t)$. However, in (r, z, t) coordinates, the resulting partial differential equation is somewhat complicated. In the following two paragraphs we consider the transformation of the invertibility principle from the physical height coordinate to a total pressure type coordinate and to the virtual potential temperature coordinate. Both of these transformations result in simpler forms of the invertibility principle.

First consider the transformation to $(r, \hat{z}, t)$ coordinates, where $\hat{z} = \hat{z}_a[1 - (p/p_0)^\kappa]$ is the pseudoheight, with $\hat{z}_a = c_{Pa}T_0/g$. With this vertical coordinate, the gradient and hydrostatic equations are $(f + v/r)v = \partial\phi/\partial r$ and $(g/T_0)\theta_\rho = \partial\phi/\partial\hat{z}$, where $\phi = gz$ is the geopotential and where the radial derivatives are now understood to be at fixed $\hat{z}$. In a similar fashion, the transformation of (26), along with the use of the gradient and hydrostatic relations, yields (27a), which can be regarded as a second-order nonlinear partial differential equation relating the geopotential ϕ to the PV. The boundary conditions for (27a) are that ϕ goes to a specified far-field geopotential $\tilde{\phi}(\hat{z})$ as $r \to \infty$ and that $\partial\phi/\partial\hat{z}$ is given in terms of a known boundary θ_ρ at the top and bottom boundaries. Thus, the complete invertibility principle in $\hat{z}$ coordinates is

$$\frac{T_0}{g\hat{\rho}}\left\{\left[f^2 + \frac{\partial}{r^3\partial r}\left(r^3\frac{\partial\phi}{\partial r}\right)\right]\frac{\partial^2\phi}{\partial\hat{z}^2} - \left(\frac{\partial^2\phi}{\partial r\partial\hat{z}}\right)^2\right\}\left(f^2 + \frac{4}{r}\frac{\partial\phi}{\partial r}\right)^{-1/2}$$

$$= P, \tag{27a}$$

$$\phi \to \tilde{\phi}(\hat{z}) \quad \text{as } r \to \infty, \tag{27b}$$

$$\frac{\partial\phi}{\partial\hat{z}} = \frac{g}{T_0}\theta_\rho \quad \text{at lower and upper boundaries,} \tag{27c}$$

where $\hat{\rho}(\hat{z}) = \rho_0(1 - \hat{z}/\hat{z}_a)^{(1-\kappa)/\kappa}$ is the pseudodensity (a known function of $\hat{z}$). The solution of the nonlinear, second-order partial differential equation (27a), with the appropriate boundary conditions (27b) and (27c), yields the geopotential $\phi(r, \hat{z}, t)$, from which $v(r, \hat{z}, t)$ and $\theta_\rho(r, \hat{z}, t)$ can be calculated using the gradient and hydrostatic relations.

Now, assuming that θ_ρ is a monotonic function of the physical height z, consider the transformation to (r, θ_ρ, t) coordinates. With this vertical coordinate, the gradient and hydrostatic equations are $(f + v/r)v = \partial M/\partial r$ and $\Pi = \partial M/\partial\theta_\rho$, where $M = c_{Pa}T_\rho + gz$ is the Montgomery potential (based on virtual temperature), $\Pi = c_{Pa}(p/p_0)^\kappa$ is the Exner function (based on total pressure), and where the radial derivatives are now understood to be at fixed θ_ρ. In a similar fashion, the transformation of (26), along with the use of the gradient and hydrostatic relations, yields (28a), which can be regarded as a second-order nonlinear partial differential equation relating the Montgomery potential M to the potential vorticity P. The boundary conditions for (28a) are that M goes to a specified far-field Montgomery potential $\tilde{M}(\theta_\rho)$ as $r \to \infty$ and that $\partial M/\partial\theta_\rho$ is given in terms of a known boundary Π at the top and bottom boundaries. Thus, the complete invertibility principle in θ_ρ coordinates is

$$-g\Gamma\left[f^2 + \frac{\partial}{r^3\partial r}\left(r^3\frac{\partial M}{\partial r}\right)\right]\left(f^2 + \frac{4}{r}\frac{\partial M}{\partial r}\right)^{-1/2}\left(\frac{\partial^2 M}{\partial\theta_\rho^2}\right)^{-1} = P, \tag{28a}$$

$$M \to \tilde{M}(\theta_\rho) \quad \text{as } r \to \infty, \tag{28b}$$

$$\frac{\partial M}{\partial\theta_\rho} = \Pi \quad \text{at lower and upper boundaries,} \tag{28c}$$

where $\Gamma(p) = d\Pi/dp = \kappa\Pi/p$. The solution of the nonlinear, second-order partial differential equation (28a), with appropriate boundary conditions (28b) and (28c), yields the Montgomery potential $M(r, \theta_\rho, t)$, from which $v(r, \theta_\rho, t)$ and $\Pi(r, \theta_\rho, t)$ can be calculated using the gradient and hydrostatic relations.

Of course, the $\hat{z}$-coordinate invertibility relation (27), the θ_ρ-coordinate invertibility relation (28), and the original relations (22)–(26) are simply different mathematical forms of the same physical principle. For the special case when all independent and dependent variables are interpreted in terms of their dry limits, the invertibility relation (27) is equivalent to the one solved by Hoskins et al. (1985) and Thorpe (1985, 1986), while (28) is equivalent to the one solved by Schubert and Alworth (1987) and Möller and Smith (1994). In their calculations these authors used a further transformation from the physical radius r to the potential radius R, which is defined by $\tfrac{1}{2}fR^2 = \tfrac{1}{2}fr^2 + rv$. Regardless of this additional transformation, we can interpret the results of these previous "dry" invertibility studies in terms of our moist model, since the dry and moist invertibility problems are isomorphic under the interchanges $\theta \leftrightarrow \theta_\rho$, $T \leftrightarrow T_\rho$, etc.

In passing we note that the invertibility principle (27), expressed in the $\hat{z}$ coordinate, or its equivalent, (28), expressed in the θ_ρ coordinate, represents only one member of a family of invertibility principles. Other members of the family are generated by replacing the gradient wind equation with different horizontal balance relations, for example, the geostrophic equation, the nonlinear balance equation, or the asymmetric balance equation (Shapiro and Montgomery 1993). In any event, the existence of such a family of invertibility principles indicates that the potential vorticity (20b), obtained through the choice $\psi = \theta_\rho$, is the form that maintains a direct connection to the balanced dynamics and is hence the appropriate moist generalization of the well-known (dry) Ertel PV. In the next section we discuss a commonly suggested choice for ψ that turns out to be unacceptable from the standpoint of possessing an invertibility principle.

5. An alternative approach to the choice of ψ

Our approach in deriving (20) has been to choose ψ in such a way as to annihilate the $\nabla\psi \cdot (\nabla\rho \times \nabla p)$ term on the right-hand side of (17). An alternative approach attempts to choose ψ in such a way as to annihilate the $\boldsymbol{\zeta} \cdot \nabla\dot{\psi}$ term on the right-hand side of (17). One way to accomplish this, at least in an approximate sense, is to note that (4) can be written in the form

$$\frac{Ds}{Dt} = \frac{1}{\rho_a}[Q_\sigma - \nabla \cdot (\sigma_r\mathbf{U})], \tag{29}$$

where $s = \sigma/\rho_a$ is the "dry-air-specific" entropy of moist air. Now consider a physical situation that warrants the neglect of the right-hand side of (29). For example, such a situation might occur when radiation and precipitation effects play a secondary role in the entropy budget. Then, defining the equivalent potential temperature by $\theta_e = T_0\exp(s/c_{Pa})$, (29) reduces to $\dot{\theta}_e \equiv D\theta_e/Dt \approx 0$ and the choice $\psi = \theta_e$ in (17) leads to

$$\frac{D}{Dt}\left(\frac{1}{\rho}\boldsymbol{\zeta}\cdot\nabla\theta_e\right)$$
$$\approx \frac{1}{\rho^3}\nabla\theta_e\cdot(\nabla\rho\times\nabla p) + \frac{1}{\rho}(\nabla\times\mathbf{F})\cdot\nabla\theta_e, \qquad (30)$$

which is sometimes referred to as the equivalent potential vorticity principle. The equivalent potential vorticity $\rho^{-1}\boldsymbol{\zeta}\cdot\nabla\theta_e$ has been used as a diagnostic in several studies, for example, in the analysis of model simulations of the rotation and propagation of supercell thunderstorms (Rotunno and Klemp 1985) and model simulations of extratropical cyclones with embedded latent heat release (Cao and Cho 1995; Persson 1995). A variant, using Hauf and Höller's (1987) entropy temperature for the choice of ψ, has been used by Rivas Soriano and García Díez (1997) to study the effects of ice.

The choice $\psi = \theta_e$ is problematic in three respects: (i) in a precipitating atmosphere with radiative forcing, the term $\rho^{-1}\boldsymbol{\zeta}\cdot\nabla\dot{\theta}_e$ is not exactly eliminated; (ii) because θ_e does not depend on ρ and p only, but also on ρ_m and ρ_r, the term $\nabla\theta_e\cdot(\nabla\rho\times\nabla p)$ is not eliminated from (30); and (iii) when attempting to set up an invertibility principle for $\rho^{-1}\boldsymbol{\zeta}\cdot\nabla\theta_e$, analogous to the invertibility principle (22)–(26) for $\rho^{-1}\boldsymbol{\zeta}\cdot\nabla\theta_\rho$, one finds that knowledge of the $\rho^{-1}\boldsymbol{\zeta}\cdot\nabla\theta_e$ field is not sufficient and that additional information on the moisture field is required for determination of the balanced wind and mass fields. In other words, $\rho^{-1}\boldsymbol{\zeta}\cdot\nabla\theta_e$ does not carry all the essential dynamical information on the balanced flow like $\rho^{-1}\boldsymbol{\zeta}\cdot\nabla\theta_\rho$ does. For balanced flow and slow manifold dynamics, the relevant part of the vector $\boldsymbol{\zeta}$ is perpendicular to the θ_ρ surfaces, not to the θ_e surfaces. We conclude that, because $\rho^{-1}\boldsymbol{\zeta}\cdot\nabla\theta_e$ is not invertible, (20) is more useful than (30) for the study of moist, precipitating, balanced flows.

6. Conclusions

The physical model that serves as the basis for the derivation of the generalized potential vorticity principle (20) is the nonhydrostatic moist model [(1)–(13)]. In this model, pressure is not used as one of the prognostic variables, since it is not a conservative property and its use as a prognostic variable would lead to an approximate treatment of moist thermodynamics. Rather, the prognostic variable for the thermodynamic state is σ, the entropy of moist air per unit volume, with temperature and total pressure (the sum of the partial pressures of dry air and water vapor) determined diagnostically. There are several unique aspects of this model that are worth noting.

1) The model dynamics are exact in the sense that there is no hydrostatic approximation and no traditional approximation (i.e., selectively replacing the actual radius by the constant radius to mean sea level and neglecting Coriolis terms proportional to the cosine of latitude); this means that the associated angular momentum and energy principles are exact.
2) The connection between dynamics and thermodynamics is through the gradient of pressure, which includes the partial pressures of dry air and water vapor.
3) The first law of thermodynamics is expressed in terms of σ, the entropy density of moist air; all the usual approximations associated with moist thermodynamics are thereby avoided.
4) There is no cumulus parameterization; the frontier of empiricism is pushed back to the microphysical parameterization of the precipitation process through $\mathbf{U}$ and Q_r.
5) The model is modular in the sense that ice can be included by specifying $E(T)$ to be synthesized from the saturation formulas over water and ice (Ooyama 1990).

Even with a nonhydrostatic moist model of this generality, it is possible to derive a PV principle that is a straightforward generalization of the well-known (dry) Ertel PV principle. Using (20b) it is possible to construct PV maps as diagnostics of the nonhydrostatic moist model. Such a PV diagnostic is a useful aid in understanding the relationship between nonhydrostatic moist convection and the large-scale balanced flow. Hausman (2001) has produced such PV cross sections for a nonhydrostatic hurricane model based on the axisymmetric, f plane versions of (1)–(13). The PV structure associated with the intense hurricane stage of the numerical simulation consists of an annular ring of very high PV (more than 200 PV units) extending through the whole troposphere between 10- and 15-km radius. In a fully three-dimensional model, such a PV structure would be subject to combined barotropic–baroclinic instability, the barotropic aspects of which were studied by Schubert et al. (1999). Hausman also compared cross sections of P, computed from (26), with cross sections of dry Ertel PV. The differences are quite small, indicating that dry Ertel PV and its invertibility principle can give an accurate description of the balanced aspects of hurricane dynamics if the frictional and moist-diabatic source/sink terms for the dry PV are accurately parameterized.

In closing we note that there is an impermeability principle associated with the flux form of (20), that is, the θ_ρ surfaces are impermeable to ρP, even though they are permeable to mass. This is discussed in appendix D.

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APPENDIX A

List of Symbols

Mass densities, temperatures, pressures, velocities

ρ_a	Mass density of dry air
ρ_v	Mass density of water vapor
ρ_c	Mass density (as aerosol) of airborne condensate (droplets or ice crystals)
ρ_r	Mass density (as aerosol) of precipitating water substance (liquid or ice)
$\rho_m = \rho_v + \rho_c$	Mass density of airborne moisture (vapor plus airborne condensate)
$\rho_{am} = \rho_a + \rho_m$	Mass density of dry air and airborne moisture
$\rho = \rho_a + \rho_m + \rho_r$	Total mass density (dry air plus airborne moisture plus precipitation)
T_1	Temperature, when condensation does not occur or is not allowed
T_2	Temperature if saturated, or wet-bulb temperature if unsaturated
$T = \max(T_1, T_2)$	Temperature
$T_\rho = p/(\rho R_a)$	Virtual temperature
$\theta_\rho = T_\rho(p_0/p)^\kappa$	Virtual potential temperature
$\theta_e = T_0 \exp(s/c_{Pa})$	Equivalent potential temperature
p_a	Partial pressure of dry air
p_v	Partial pressure of water vapor
$p = p_a + p_v$	Total pressure of moist air
$\mathbf{u}$	Velocity of dry air and airborne moisture (relative to earth)
$\mathbf{u}_r$	Velocity of precipitation (relative to earth)
$\mathbf{U} = \mathbf{u}_r - \mathbf{u}$	Velocity of precipitation (relative to dry air and airborne moisture)
$\overline{\mathbf{u}} = [(\rho_a + \rho_m)\mathbf{u} + \rho_r\mathbf{u}_r]/\rho$	Density-weighted-mean velocity

Specific entropies (J kg^{-1} K^{-1}) and entropy densities (J m^{-3} K^{-1})

s_a	Specific entropy of dry air, defined by $s = c_{Va}\ln(T/T_0) - R_a \ln(\rho_a/\rho_{a0})$
$s_m^{(1)}$	Specific entropy of airborne moisture in state 1, defined by $s_m^{(1)} = c_{Vv}\ln(T/T_0) - R_v \ln(\rho_m/\rho_{m0}^*) + \Lambda_0$
$s_m^{(2)}$	Specific entropy of airborne moisture in state 2, defined by $s_m^{(2)} = C(T) + D(T)/\rho_m$
s_r	Specific entropy of condensed water (cloud or precipitation)
$s = \sigma/\rho_a$	Dry-air-specific entropy of moist air
$\sigma_a = \rho_a s_a$	Entropy density of dry air
$\sigma_m = \rho_m s_m^{(1)}$	Entropy density of airborne water substance for state 1
$\sigma_m = \rho_m s_m^{(2)}$	Entropy density of airborne water substance for state 2
$\sigma_r = \rho_r s_r$	Entropy density of precipitating water substance
$\sigma = \sigma_a + \sigma_m + \sigma_r$	Total entropy density
$S_1(\rho_a, \rho_m, T)$	Entropy density function for state 1, defined by $S_1(\rho_a, \rho_m, T) = \rho_a s_a + \rho_m s_m^{(1)}$
$S_2(\rho_a, \rho_m, T)$	Entropy density function for state 2, defined by $S_2(\rho_a, \rho_m, T) = \rho_a s_a + \rho_m s_m^{(2)}$

Constants

Ω	Angular rotation rate of the Earth
R_a	Gas constant of dry air
R_v	Gas constant of water vapor
c_{Va}	Specific heat of dry air at constant volume
c_{Vv}	Specific heat of water vapor at constant volume
$c_{Pa} = c_{Va} + R_a$	Specific heat of dry air at constant pressure
$c_{Pv} = c_{Vv} + R_v$	Specific heat of water vapor at constant pressure
$\kappa = R_a/c_{Pa}$	
p_0	Reference pressure, 100 kPa
T_0	Reference temperature, 273.15 K
$\rho_0 = p_0/(R_a T_0)$	Reference density for dry air
$E_0 = E(T_0)$	Saturation vapor pressure at T_0
$\rho_0^* = \rho_v^*(T_0)$	Mass density of saturated vapor at T_0
$\Lambda_0 = \Lambda(T_0)$	Gain of entropy by evaporating a unit mass of water at T_0
$\hat{z}_a = c_{Pa}T_0/g$	Pseudoheight at which $p = 0$

Defined functions of temperature

$\Lambda(T) = R_v T(d \ln E(T)/dT)$	Gain of entropy by evaporating a unit mass of water at T
$C(T)$	Entropy of a unit mass of condensate at T as measured from the reference state T_0
$D(T) = dE(T)/dT$	Gain of entropy per unit volume by evaporating a sufficient amount of water, $\rho_*(T)$, to saturate the volume at T
$E(T)$	Saturation vapor pressure, which may be synthesized from the saturation vapor pressures over water and ice
$\rho_v^*(T) = E(T)/(R_v T)$	Mass density of saturated vapor

Others

$\mathbf{F}_{am}$	Frictional force acting on ρ_{am}
$\mathbf{F}_r$	Vertical drag force acting on ρ_r
$\overline{\mathbf{F}}$	Total frictional force per unit mass (including precipitation)
$\overline{\mathbf{F}} = \mathbf{F} - (\rho_r/\rho)\mathbf{U} \times \boldsymbol{\zeta}$	Frictional force appearing in (22)
$\boldsymbol{\zeta} = 2\boldsymbol{\Omega} + \boldsymbol{\nabla} \times \mathbf{u}$	Absolute vorticity vector
$\mathbf{j} = \boldsymbol{\nabla}\theta_\rho/\lvert\boldsymbol{\nabla}\theta_\rho\rvert$	Unit vector normal to θ_ρ surface
$\mathbf{k} = \boldsymbol{\zeta}/\lvert\boldsymbol{\zeta}\rvert$	Unit vector pointing along absolute vorticity vector
$\underline{D}/Dt = \partial/\partial t + \mathbf{u}\cdot\boldsymbol{\nabla}$	Derivative following the dry air, water vapor, and airborne condensate
$\overline{D}/Dt = \partial/\partial t + \overline{\mathbf{u}}\cdot\boldsymbol{\nabla}$	Derivative following the density-weighted-mean velocity $\overline{\mathbf{u}}$
$D^{(r)}/Dt = \partial/\partial t + \mathbf{u}_r\cdot\boldsymbol{\nabla}$	Derivative following the precipitation
$\underline{\dot{\theta}}_\rho = \underline{D}\theta_\rho/Dt$	Diabatic source term in (20a)
$\overline{\dot{\theta}}_\rho = \overline{D}\theta_\rho/Dt$	Diabatic source term in (22)
Q_r	Conversion rate of ρ_m to ρ_r
Q_σ	Entropy source term (e.g., radiation)
Φ	Potential for sum of Newtonian gravitational force and centrifugal force
$\phi = gz$	Geopotential
$M = c_{Pa}T_\rho + gz$	Montgomery potential (based on virtual temperature)
$\Pi = c_{Pa}(p/p_0)^\kappa$	Exner function (based on total pressure)
$\Gamma(p) = d\Pi/dp = \kappa\Pi/p$	Derivative of the Exner function with respect to p
$\hat{z} = \hat{z}_a[1 - (p/p_0)^\kappa]$	Pseudo-height
$\hat{\rho}(\hat{z}) = \rho_0(1 - \hat{z}/\hat{z}_a)^{(1-\kappa)/\kappa}$	Pseudo-density (a known function of $\hat{z}$)

APPENDIX B

Exact and Approximate Momentum Equations

To derive the single predictive equation for $\mathbf{u}$, first consider separately the momentum equations for $\rho_{am} = \rho_a + \rho_m$ and ρ_r, which are

$$\frac{\partial(\rho_{am}\mathbf{u})}{\partial t} + \boldsymbol{\nabla}\cdot(\rho_{am}\mathbf{u}\mathbf{u}) + \rho_{am}2\boldsymbol{\Omega}\times\mathbf{u} + \rho_{am}\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}p = \rho_{am}\mathbf{F}_{am} - \rho_r\mathbf{F}_r - Q_r\mathbf{u}, \tag{B1}$$

$$\frac{\partial(\rho_r\mathbf{u}_r)}{\partial t} + \boldsymbol{\nabla}\cdot(\rho_r\mathbf{u}_r\mathbf{u}_r) + \rho_r 2\boldsymbol{\Omega}\times\mathbf{u}_r + \rho_r\boldsymbol{\nabla}\Phi = \rho_r\mathbf{F}_r + Q_r\mathbf{u}_r, \tag{B2}$$

where $\mathbf{F}_{am}$ is the frictional force acting on ρ_{am} and $\mathbf{F}_r$ is the vertical drag force acting on ρ_r. Note that there is no pressure gradient force in (B2) since the fractional volume of precipitation is assumed to be negligibly small. As discussed by Ooyama (2001), there is a serious contradiction in using both (B1) and (B2), since prediction of both $\mathbf{u}$ and $\mathbf{u}_r$ is equivalent to prediction of $\mathbf{U}$, which we are assuming is a diagnostic variable. Ooyama has suggested using an approximation of the sum of (B1) and (B2). The exact sum of (B1) and (B2), when converted to advective form, is

$$\frac{D\mathbf{u}}{Dt} + \frac{\rho_r}{\rho}\left(\frac{D^{(r)}\mathbf{U}}{Dt} + 2\boldsymbol{\Omega}\times\mathbf{U}\right) + 2\boldsymbol{\Omega}\times\mathbf{u} + \boldsymbol{\nabla}\Phi + \frac{1}{\rho}\boldsymbol{\nabla}p = \mathbf{F}, \tag{B3}$$

where

$$\mathbf{F} = \frac{1}{\rho}[\rho_{am}\mathbf{F}_{am} - \rho_r(\mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{u}], \tag{B4}$$

and where $D/Dt = \partial/\partial t + \mathbf{u}\cdot\boldsymbol{\nabla}$ and $D^{(r)}/Dt = \partial/\partial t + \mathbf{u}_r\cdot\boldsymbol{\nabla}$. Neglect of the $(\rho_r/\rho)(D^{(r)}\mathbf{U}/Dt + 2\boldsymbol{\Omega}\times\mathbf{U})$ term in (B3) results in

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega}\times\mathbf{u} + \boldsymbol{\nabla}\Phi + \frac{1}{\rho}\boldsymbol{\nabla}p = \mathbf{F}, \tag{B5}$$

which is the approximate momentum equation used by Ooyama (2001) in his numerical simulations.[B1] We shall use this same approximate momentum equation in our

[B1] In Ooyama's numerical simulations, $\mathbf{F}_{am} = 0$ and $\boldsymbol{\Omega}$ is antiparallel with $\mathbf{U}$ so that $2\boldsymbol{\Omega}\times\mathbf{U} = 0$.

derivation of the PV principle. For the PV derivation, it is most convenient to put this approximate momentum equation in a rotational form. Thus, using $(\mathbf{u} \cdot \nabla)\mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u})$, we can convert the advective form (B5) to the rotational form

$$\frac{\partial \mathbf{u}}{\partial t} + (2\boldsymbol{\Omega} + \nabla \times \mathbf{u}) \times \mathbf{u} + \nabla\left(\frac{1}{2}\mathbf{u} \cdot \mathbf{u} + \Phi\right) + \frac{1}{\rho}\nabla p = \mathbf{F}. \tag{B6}$$

In passing we note that the particular approximation (B5) is not required for the derivation of the PV principle. In fact, the unapproximated sum of (B1) and (B2), written as a prognostic equation for the density-weighted-mean velocity $\overline{\mathbf{u}} = \mathbf{u} + (\rho_r/\rho)\mathbf{U}$, can be used as the starting point for the derivation of the "exact" potential vorticity principle. The resulting exact PV principle (Hausman 2001) takes the form of (C1), but with slight modifications (e.g., $\overline{\boldsymbol{\zeta}} = 2\boldsymbol{\Omega} + \nabla \times \overline{\mathbf{u}}$ replaces $\boldsymbol{\zeta} = 2\boldsymbol{\Omega} + \nabla \times \mathbf{u}$ in the definition of P). However, it is important to note that the difference between the density-weighted-mean velocity $\overline{\mathbf{u}}$ and the velocity $\mathbf{u}$ tends to be quite small. In fact, the two are identical in nonprecipitating regions (where $\rho_r = 0$), and their horizontal component is identical in precipitating regions. Even in heavily precipitating regions with rainfall rates of 36 mm h^{-1}, the magnitude of $(\rho_r/\rho)\mathbf{U}$ is only 0.01 m s^{-1}, so that the vertical component of the density-weighted-mean velocity is approximately 0.01 m s^{-1} smaller than the vertical component of the dry air velocity. In addition, although it has a certain theoretical appeal, the use of the density-weighted-mean velocity presents practical difficulties both in observational analysis and in numerical modeling. Observationally, ρ_r and the vertical component of $\mathbf{u}$ are both difficult to measure, so it may be impossible in practice to make a meaningful distinction between $\overline{\mathbf{u}}$ and $\mathbf{u}$. In model results, ρ_r and the vertical component of $\mathbf{u}$ are both highly dependent on model resolution. The use of $\overline{\mathbf{u}}$ as a prognostic variable is also problematic in numerical schemes that require a priori specification of the lower boundary condition. For example, with a flat lower boundary the vertical component of the dry air velocity $\mathbf{u}$ is zero but the vertical component of $\overline{\mathbf{u}}$ is not zero, and in fact changing, in precipitating regions where condensed water substance is leaving the atmospheric model domain. Thus, in both observational analysis and model output diagnostics, compromising assumptions on the "exact theory" seem inevitable. For this reason, we have chosen to introduce the plausible and consistent approximation (B5) at the start of the PV derivation.

APPENDIX C

An Alternative Form of the PV Principle (20a)

If the effects of precipitation contained in the last term of (20a) are distributed over the other three terms, we obtain an equation that is formally similar to the one usually given for the dry case. To accomplish this we first define the density-weighted-mean velocity as $\overline{\mathbf{u}} = \mathbf{u} + (\rho_r/\rho)\mathbf{U}$ and the derivative following this velocity as $\overline{D}/Dt = \partial/\partial t + \overline{\mathbf{u}} \cdot \nabla$. We then rewrite (20a) as

$$\begin{aligned}\frac{\overline{D}P}{Dt} &= \frac{1}{\rho}\nabla \cdot (\mathbf{F} \times \nabla\theta_\rho + \boldsymbol{\zeta}\dot{\theta}_\rho + \rho_r \mathbf{U}P) \\ &= \frac{1}{\rho}\nabla \cdot (\overline{\mathbf{F}} \times \nabla\theta_\rho + \boldsymbol{\zeta}\overline{\dot{\theta}}_\rho) \\ &= \frac{1}{\rho}(\nabla \times \overline{\mathbf{F}}) \cdot \nabla\theta_\rho + \frac{1}{\rho}\boldsymbol{\zeta} \cdot \nabla\overline{\dot{\theta}}_\rho \\ &= P\left(\frac{\mathbf{j} \cdot \nabla \times \overline{\mathbf{F}}}{\mathbf{j} \cdot \boldsymbol{\zeta}} + \frac{\mathbf{k} \cdot \nabla\overline{\dot{\theta}}_\rho}{\mathbf{k} \cdot \nabla\theta_\rho}\right),\end{aligned} \tag{C1}$$

where $\overline{\mathbf{F}} = \mathbf{F} - (\rho_r/\rho)\mathbf{U} \times \boldsymbol{\zeta}$, $\overline{\dot{\theta}}_\rho = \overline{D}\theta_\rho/Dt$, and in going from the first line to the second line we have used the triple vector product $\rho_r\mathbf{U}P = (\rho_r/\rho)\mathbf{U}(\boldsymbol{\zeta} \cdot \nabla\theta_\rho) = \boldsymbol{\zeta}[(\rho_r/\rho)\mathbf{U} \cdot \nabla\theta_\rho] + \nabla\theta_\rho \times [(\rho_r/\rho)\mathbf{U} \times \boldsymbol{\zeta}]$. The forms given on the first two lines of (C1) are convenient because they express the total source term for $\overline{D}P/Dt$ as the divergence of a single vector field, while the forms given on the last two lines are convenient because of their formal similarity to the conventional forms for the dry case. Note that the last line of (C1) is an alternative form to (21).

APPENDIX D

Impermeability Principle

It is natural to ask if there is an impermeability principle associated with (20), that is to ask if the θ_ρ surfaces are impermeable to ρP even though they are permeable to mass. Indeed there does exist such an impermeability principle, and its existence does not depend on the momentum equation or on the choice $\psi = \theta_\rho$. To see this, let us return to (14) and recall the discussion given by Haynes and McIntyre (1987, 1990). Surfaces of constant ψ move through space, and an observer moving with the velocity $\mathbf{u}$ crosses these surfaces, since $\dot{\psi} \equiv D\psi/Dt = \partial\psi/\partial t + \mathbf{u} \cdot \nabla\psi \neq 0$. We can regard the ψ surfaces as being carried through space by a hypothetical velocity field $\hat{\mathbf{u}}$, which satisfies $\partial\psi/\partial t + \hat{\mathbf{u}} \cdot \nabla\psi = 0$. Thus, an observer moving with the velocity field $\mathbf{u}$ will cross ψ surfaces while an observer moving with the hypothetical velocity field $\hat{\mathbf{u}}$ will remain on the same ψ surface. The flux in (14) can thus be written as $\nabla\psi \times (\partial\mathbf{u}/\partial t) + \boldsymbol{\zeta}(\hat{\mathbf{u}} \cdot \nabla\psi)$. Since the triple vector product rule yields $\boldsymbol{\zeta}(\hat{\mathbf{u}} \cdot \nabla\psi) = \hat{\mathbf{u}}(\boldsymbol{\zeta} \cdot \nabla\psi) + \nabla\psi \times (\boldsymbol{\zeta} \times \hat{\mathbf{u}})$, we can write (14) as

$$\frac{\partial}{\partial t}(\boldsymbol{\zeta} \cdot \nabla\psi) + \nabla \cdot \left[\hat{\mathbf{u}}(\boldsymbol{\zeta} \cdot \nabla\psi) + \nabla\psi \times \left(\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\zeta} \times \hat{\mathbf{u}}\right)\right] = 0, \tag{D1}$$

which has a form analogous to (15). The $\nabla\psi \times (\partial\mathbf{u}/\partial t + \boldsymbol{\zeta} \times \hat{\mathbf{u}})$ part of the flux is perpendicular to $\nabla\psi$, that is, along the ψ surface. The $\hat{\mathbf{u}}(\boldsymbol{\zeta} \cdot \nabla\psi)$ part of the flux is not necessarily along the ψ surface, but the associated velocity is $\hat{\mathbf{u}}$, and an observer moving with velocity $\hat{\mathbf{u}}$ remains on the same ψ surface. Thus, it is impossible to flux $\boldsymbol{\zeta} \cdot \nabla\psi$ across a ψ surface. In summary, the ψ surface is impermeable to $\boldsymbol{\zeta} \cdot \nabla\psi$, regardless of how the arbitrary scalar ψ is ultimately defined. In particular, when we make the choice $\psi = \theta_\rho$, we can say that the θ_ρ surfaces are impermeable to ρP as defined by (20b).

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