Vorticity Coordinates, Transformed Primitive Equations, and a Canonical Form for Balanced Models

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ABSTRACT

A potential pseudodensity principle is derived for the quasi-static primitive equations on the sphere. An important step in the derivation of this principle is the introduction of "vorticity coordinates"—that is, new coordinates whose Jacobian with respect to the original spherical coordinates is the dimensionless absolute isentropic vorticity. The vorticity coordinates are closely related to Clebsch variables and are the primitive equation generalizations of the geostrophic coordinates used in semigeostrophic theory. The vorticity coordinates can be used to transform the primitive equations into a canonical form. This form is mathematically similar to the geostrophic relation. There is flexibility in the choice of the potential function appearing in the canonical momentum equations. This flexibility can be used to force the vorticity coordinates to move with some desired velocity, which results in an associated simplification of the material derivative operator. The end result is analogous to the way ageostrophic motions become implicit when geostrophic coordinates are used in semigeostrophic theory.

1. Introduction

Quasigeostrophic theory can be mathematically formulated as two equations—a predictive equation for the quasigeostrophic potential vorticity and a diagnostic invertibility relation to obtain the balanced wind and mass fields from the quasigeostrophic potential vorticity. In the predictive equation there is no vertical advection and the horizontal advection is geostrophic. This simplicity of the quasigeostrophic formulation, combined with the fact that the invertibility relation contains a linear elliptic operator, has proven valuable in establishing a powerful theoretical formalism for the quasigeostrophic system.

Semigeostrophic theory lies closer to the primitive equations than does quasigeostrophic theory. It includes horizontal ageostrophic advection, vertical advection, twisting, and nonlinear stretching effects in the vorticity equation and is thus able to yield fairly accurate simulations of fronts and jets. Even with these additional effects, semigeostrophic theory, if appropriately formulated, can also be written as a closed system of two equations—a predictive equation for the potential pseudodensity and a diagnostic invertibility relation to obtain the associated balanced wind and mass fields [see McWilliams and Gent (1980) for a theoretical discussion; Schubert et al. (1989) and Fulton and Schubert (1991) for applications of the f-plane theory; Salmon (1983, 1985, 1988a,b), Shutts (1989), and Magnusdottir and Schubert (1990, 1991) for the β-plane and spherical cases]. The term "appropriately formulated" is equivalent to saying that the combined use of isentropic and geostrophic coordinates is essential to making both the vertical advection and the horizontal ageostrophic advection implicit. Only then can semigeostrophic theory take essentially the same mathematical form as quasigeostrophic theory. The predictive variable is potential pseudodensity, which is inversely proportional to potential vorticity. On the f plane the potential pseudodensity is simply equal to f (the Coriolis parameter) divided by the potential vorticity. In contrast, on both the β plane and the sphere, the geostrophic coordinates enter the definition of the potential pseudodensity since in these cases the Coriolis parameter is evaluated at the geostrophic latitude.

Although the above semigeostrophic theories are applicable to a variety of problems such as fronts, jets, occluding baroclinic waves, and sudden stratospheric warmings, there are many problems for which the theories are deficient. For example, in tropical cyclones the curvature vorticity is as important as the shear vor-

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ticity, and the balance is gradient rather than geostrophic. However, even in this case, it is possible to formulate an axisymmetric gradient balanced model using combined isentropic and potential radius coordinates (Schubert and Alworth 1987). This theory also reduces to a predictive equation for potential pseudodensity and an associated invertibility relation. Another example is the zonally symmetric, balanced model of the ITCZ and the Hadley circulation, in which case the use of combined isentropic and potential latitude coordinates leads to the same mathematical structure (Schubert et al. 1991).

This raises the question, Can we formulate a three-dimensional generalized global balanced theory[1] more accurate than semigeostrophic theory, but which retains essentially the same mathematical structure of one predictive equation for potential pseudodensity and one diagnostic invertibility relation? Obviously, we must formulate an approximation more general than the geostrophic momentum approximation and define transformed coordinates more general than geostrophic coordinates. The present paper is a contribution to this general line of research. In particular, we will discuss the transformation of the primitive equations to vorticity coordinates, which makes the advecting velocities look formally geostrophic, and the derivation of a general potential pseudodensity principle. The expression of the primitive equations in terms of vorticity coordinates naturally introduces the potential pseudodensity equation as one of four predictive equations. The other three predictive equations are the two horizontal Lagrangian particle displacement equations and an equation for the evolution of the first Clebsch potential. We will explore the advantages introduced into the primitive equations by writing them in vorticity coordinates. One of the principal advantages of the transformed set is its close connection to the balanced theories discussed above. Finally, we will discuss a general structure for balanced theories.

The derivations are first presented for the shallow-water equations on an f plane (section 2) and then for the fully stratified, quasi-static primitive equations on the sphere (section 3).

2. The shallow-water equations on an f plane

a. Vorticity coordinate transformation of the shallow-water equations

The shallow-water equations on an f plane can be written

$$\frac{\partial u}{\partial t} - \zeta v + \frac{\partial}{\partial x}\left[gh + \frac{1}{2}(u^2 + v^2)\right] = 0, \quad (2.1)$$

$$\frac{\partial v}{\partial t} + \zeta u + \frac{\partial}{\partial y}\left[gh + \frac{1}{2}(u^2 + v^2)\right] = 0, \quad (2.2)$$

$$\frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \quad (2.3)$$

where u and v are the eastward and northward components of the velocity, h is the fluid depth,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \quad (2.4)$$

the total derivative, and

$$\zeta = f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad (2.5)$$

the absolute vorticity.

Now consider a transformation from the coordinates (x, y, t) to the new coordinates $(X, Y, \mathscr{T})$, where $\mathscr{T} = t$. The symbol $\mathscr{T}$ has been introduced to distinguish the time derivative at fixed (X, Y) from the time derivative at fixed (x, y). We require the new coordinates to be vorticity coordinates in the sense that the Jacobian of (X, Y) with respect to (x, y) is the dimensionless absolute vorticity; that is,

$$\zeta = f\frac{\partial(X, Y)}{\partial(x, y)}. \quad (2.6)$$

If we eliminate ζ between (2.5) and (2.6), the resulting expression can be rearranged into the form

$$\frac{\partial}{\partial x}\left\{v - \frac{1}{2}f\left[(X - x)\left(\frac{\partial Y}{\partial y} + 1\right) - (Y - y)\frac{\partial X}{\partial y}\right]\right\} - \frac{\partial}{\partial y}\left\{u - \frac{1}{2}f\left[(X - x)\frac{\partial Y}{\partial x} - (Y - y)\left(\frac{\partial X}{\partial x} + 1\right)\right]\right\} = 0. \quad (2.7)$$

Thus, the first term in braces can be expressed as $\partial\chi/\partial y$ and the second term in braces by $\partial\chi/\partial x$, where $\chi(x, y, t)$ is a scalar potential. This results in

$$u = \frac{\partial \chi}{\partial x} + \frac{1}{2}f\left[(X - x)\frac{\partial Y}{\partial x} - (Y - y)\left(\frac{\partial X}{\partial x} + 1\right)\right], \quad (2.8)$$

$$v = \frac{\partial \chi}{\partial y} + \frac{1}{2}f\left[(X - x)\left(\frac{\partial Y}{\partial y} + 1\right) - (Y - y)\frac{\partial X}{\partial y}\right]. \quad (2.9)$$

We can regard (2.8) and (2.9) as Clebsch representations of the velocity field (Lamb 1932, p. 248; Seliger

[1] Here we use the term "balanced theory" in a generic sense, that is, for a broad class of theories that filter transient gravity waves. The particular "nonlinear balance models" studied by Lorenz (1960), Charney (1962), and Gent and McWilliams (1983a,b, 1984) are members of this class.

and Whitham 1968) or as generalizations of the geostrophic coordinates. The latter interpretation results from noting that if $u - \partial\chi/\partial x$ and $v - \partial\chi/\partial y$ are approximated by their respective geostrophic wind components, and if $\partial Y/\partial x \approx 0$, $\partial X/\partial x \approx 1$, $\partial Y/\partial y \approx 1$, and $\partial X/\partial y \approx 0$, (2.8) and (2.9) in fact reduce to the geostrophic coordinates $X = x + v_g/f$ and $Y = y - u_g/f$. This special case will be discussed further in section 2c.

To transform the original momentum equations we now take $\partial/\partial t$ of (2.8) and (2.9) to obtain

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[gh + \frac{1}{2}(u^2 + v^2)\right] = f\frac{\partial(X,Y)}{\partial(t,x)} + g\frac{\partial \mathscr{H}}{\partial x}, \quad (2.10)$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial y}\left[gh + \frac{1}{2}(u^2 + v^2)\right] = f\frac{\partial(X,Y)}{\partial(t,y)} + g\frac{\partial \mathscr{H}}{\partial y}, \quad (2.11)$$

where

$$g\mathscr{H} = gh + \frac{1}{2}(u^2+v^2) + \frac{\partial\chi}{\partial t} + \frac{1}{2}f\left[(X-x)\frac{\partial Y}{\partial t} - (Y-y)\frac{\partial X}{\partial t}\right]. \quad (2.12)$$

Adding $-\zeta v$ to both sides of (2.10) and ζu to both sides of (2.11), then using (2.6) and the original momentum equations (2.1) and (2.2), we obtain

$$f\left(\frac{\partial Y}{\partial x}\frac{DX}{Dt} - \frac{\partial X}{\partial x}\frac{DY}{Dt}\right) + g\frac{\partial \mathscr{H}}{\partial x} = 0, \quad (2.13)$$

$$f\left(\frac{\partial Y}{\partial y}\frac{DX}{Dt} - \frac{\partial X}{\partial y}\frac{DY}{Dt}\right) + g\frac{\partial \mathscr{H}}{\partial y} = 0. \quad (2.14)$$

Together (2.13) and (2.14) imply that

$$U \equiv \frac{DX}{Dt} = -\frac{g}{f}\frac{\partial \mathscr{H}}{\partial Y}, \quad (2.15)$$

$$V \equiv \frac{DY}{Dt} = \frac{g}{f}\frac{\partial \mathscr{H}}{\partial X}. \quad (2.16)$$

Equation (2.15) has been obtained by eliminating DY/Dt between (2.13) and (2.14), and (2.16) by eliminating DX/Dt between (2.13) and (2.14). Alternatively, one can verify (2.15) and (2.16) by substituting them into (2.13) and (2.14), and then noting that the chain rule yields $\partial\mathscr{H}/\partial x = (\partial\mathscr{H}/\partial X)(\partial X/\partial x) + (\partial\mathscr{H}/\partial Y)(\partial Y/\partial x)$ and $\partial\mathscr{H}/\partial y = (\partial\mathscr{H}/\partial X)(\partial X/\partial y) + (\partial\mathscr{H}/\partial Y)(\partial Y/\partial y)$. Equations (2.15) and (2.16) are the canonical shallow-water equations. The total time derivative, (2.4), can be written in vorticity coordinates as

$$\frac{D}{Dt} = \frac{\partial}{\partial \mathscr{T}} + U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y}. \quad (2.17)$$

The advantage of (2.17) over (2.4) is that the horizontal advecting velocity is expressed in terms of derivatives of $\mathscr{H}$ by (2.15) and (2.16), which are mathematically analogous to the geostrophic formulas.

To summarize the above discussion: (2.8) and (2.9) constitute the Clebsch representation of the velocity field; alternatively, they can be thought of as definitions of new coordinates that allow us to write the momentum equations in canonical form, resulting in simplifications to the total derivative operator.

The governing equation for the absolute vorticity can be derived from (2.15) and (2.16) or, in the usual way, from (2.1) and (2.2). It takes the form

$$\frac{D\zeta}{Dt} + \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \quad (2.18)$$

Eliminating the divergence between (2.3) and (2.18) we obtain

$$\frac{Dh^*}{Dt} = 0, \quad (2.19)$$

where

$$h^* = \left(\frac{f}{\zeta}\right)h \quad (2.20)$$

is the potential thickness (or inverse potential vorticity). The potential thickness is simply the thickness a fluid column would acquire if the absolute vorticity ζ were changed to the constant reference value f.

b. Summary of the shallow-water equations in vorticity coordinates

We can now summarize the previous analysis as follows. In vorticity coordinates the independent variables are $(X, Y, \mathscr{T})$ and the transformed governing equations are

$$h = \left(\frac{\partial(x,y)}{\partial(X,Y)}\right)^{-1} h^*, \quad (2.21)$$

$$u = \frac{h}{h^*}\left\{-\left[\frac{\partial\chi}{\partial Y} + \frac{1}{2}f(X-x)\right]\frac{\partial y}{\partial X} + \left[\frac{\partial\chi}{\partial X} - \frac{1}{2}f(Y-y)\right]\frac{\partial y}{\partial Y}\right\} - \frac{1}{2}f(Y-y), \quad (2.22)$$

$$v = \frac{h}{h^*}\left\{\left[\frac{\partial \chi}{\partial Y} + \frac{1}{2}f(X - x)\right]\frac{\partial x}{\partial X} - \left[\frac{\partial \chi}{\partial X} - \frac{1}{2}f(Y - y)\right]\frac{\partial x}{\partial Y}\right\} + \frac{1}{2}f(X - x), \tag{2.23}$$

$$U = -\frac{g}{f}\frac{\partial \mathscr{H}}{\partial Y}, \tag{2.24}$$

$$V = \frac{g}{f}\frac{\partial \mathscr{H}}{\partial X}, \tag{2.25}$$

$$\frac{D\chi}{Dt} = g(\mathscr{H} - h) + \frac{1}{2}(u^2 + v^2) - \frac{1}{2}f[(X - x)(V + v) - (Y - y)(U + u)], \tag{2.26}$$

$$\frac{Dx}{Dt} = u, \tag{2.27}$$

$$\frac{Dy}{Dt} = v, \tag{2.28}$$

$$\frac{Dh^*}{Dt} = 0. \tag{2.29}$$

Equations (2.22) and (2.23) are simply the vorticity coordinate versions of (2.8) and (2.9). To verify that the χ terms in (2.22) come from the first term on the right-hand side of (2.8) we make use of (2.21) to obtain $\partial\chi/\partial x = \partial(\chi, y)/\partial(x, y) = (h/h^*)[\partial(x, y)/\partial(X, Y)][\partial(\chi, y)/\partial(x, y)] = (h/h^*)\partial(\chi, y)/\partial(X, Y)$. In a similar fashion we can show that $\partial Y/\partial x = -(h/h^*)\partial y/\partial X$ and $\partial X/\partial x = (h/h^*)\partial y/\partial Y$. Together, these three results confirm that (2.8) transforms to (2.22). The transformation of (2.9) to (2.23) proceeds in a similar manner. Equation (2.26) is derived by combining (2.8), (2.9), and (2.12). Taken together, (2.21)–(2.29) constitute a system of nine equations for the six diagnostic variables h, u, v, $\mathscr{H}$, U, V and the four prognostic variables χ, x, y, h^*, with the total derivative given in $(X, Y, \mathscr{T})$ space by (2.17). Obviously, an additional relation is required. If the additional relation is simply a definition of $\mathscr{H}$, and if this definition is inserted between (2.23) and (2.24), then the time evolution of the prognostic fields χ, x, y, h^* can be found by sequential calculations in the order given.

In fact, we have some freedom in choosing the additional relation. For example, the choice $\mathscr{H} = 0$ leads to the conclusion that $DX/Dt = U = 0$ and $DY/Dt = V = 0$; that is, the vorticity coordinates move with the flow so that $D/Dt = \partial/\partial\mathscr{T}$. In addition, the right-hand side of (2.26) is simplified and we now have three diagnostic equations, in addition to the four prognostic equations.

Another choice is $g\mathscr{H} = gh + \frac{1}{2}(u^2 + v^2)$. Then, if we add u times (2.1) to v times (2.2), we obtain $D\mathscr{H}/Dt = \partial h/\partial t$. If we apply (2.17) to $\mathscr{H}$ and use the canonical equations (2.24) and (2.25), we obtain $D\mathscr{H}/Dt = \partial\mathscr{H}/\partial\mathscr{T}$, so that $\partial\mathscr{H}/\partial\mathscr{T} = \partial h/\partial t$; that is, the local time change of the Bernoulli depth in (X, Y) space is equal to the local time change of the physical depth in (x, y) space.

In passing, we should keep in mind three interesting features of the transformed system (2.21)–(2.29). First of all, the system consists of four predictive equations rather than three, as in the original shallow-water equations (2.1)–(2.3). The additional prognostic equation is due to the fact that (2.27) and (2.28) explicitly determine coordinate trajectories. This trajectory information is not directly available from the solutions of the Eulerian equations (2.1)–(2.3), but is of course implicit in the form of the material derivative (2.4). The second interesting feature of (2.21)–(2.29) is the natural way in which the predictive equation for potential thickness emerges. The third interesting feature is the freedom in choosing how the vorticity coordinates move. This last feature is useful in the derivation of balanced models through approximation of the primitive set (2.21)–(2.29). One such balanced model is the semigeostrophic model, which we now discuss.

c. Reduction to semigeostrophic theory

Let us now discuss the approximations of (2.21)–(2.29) that lead to the geostrophic momentum equations and semigeostrophic theory. Recalling our freedom in the choice of $\mathscr{H}$, let us choose $g\mathscr{H} = gh + \frac{1}{2}(u_g^2 + v_g^2)$, where $(u_g, v_g) = (g/f)(-\partial h/\partial y, \partial h/\partial x)$ are the geostrophic wind components. Let us approximate (2.22) and (2.23) by $(u_g, v_g) = f(y - Y, X - x)$, which, along with the choice of $\mathscr{H}$, allow us to express the geostrophic wind as $(u_g, v_g) = (U, V) \equiv (g/f)(-\partial\mathscr{H}/\partial Y, \partial\mathscr{H}/\partial X)$. When $X = x + v_g/f$ is used in the canonical equation (2.15) and $Y = y - u_g/f$ is used in the canonical equation (2.16), we obtain $Du_g/Dt - fv + g\partial h/\partial x = 0$ and $Dv_g/Dt + fu + g\partial h/\partial y = 0$, the equations of the geostrophic momentum approximation. This suggests that other approximate dynamical systems can be obtained by leaving the canonical equations (2.15) and (2.16) unchanged but approximating the Clebsch representations (2.22) and (2.23). The restrictions on the form of these approximations as well as the form of a suitable definition of $\mathscr{H}$ are the subject of current research. An important point to make is that the above approximation procedure guarantees potential vorticity conservation. To see this, note that, as long as the canonical equations (2.15)–(2.16) are not distorted, the absolute vorticity equation (2.18) is unchanged. Then, as long as the mass continuity equation (2.3) is not distorted, the potential vorticity principle is unchanged. We will discuss approximate systems further in sections 3b and 3c.

The most succinct way of expressing semigeostrophic theory is as a prognostic equation for h^* and a diagnostic equation or invertibility principle to obtain $\mathscr{H}$ from h^*, with both relations expressed in $(X, Y, \mathscr{T})$ space. The invertibility relation is obtained by substituting $x = X - (g/f^2)\mathscr{H}_X$, $y = Y - (g/f^2)\mathscr{H}_Y$ and $h = \mathscr{H} - (g/2f^2)(\mathscr{H}_X^2 + \mathscr{H}_Y^2)$ into (2.21), resulting in

$$\frac{1}{f^4}\left\{\left(f^2 - g\frac{\partial^2 \mathscr{H}}{\partial X^2}\right)\left(f^2 - g\frac{\partial^2 \mathscr{H}}{\partial Y^2}\right) - \left(g\frac{\partial^2 \mathscr{H}}{\partial X \partial Y}\right)^2\right\} \times \left\{\mathscr{H} - \frac{g}{2f^2}\left[\left(\frac{\partial \mathscr{H}}{\partial X}\right)^2 + \left(\frac{\partial \mathscr{H}}{\partial Y}\right)^2\right]\right\} = h^*. \tag{2.30}$$

The predictive equation (2.29) can also be written as

$$\frac{\partial h^*}{\partial \mathscr{T}} = \frac{g}{f}\frac{\partial(h^*, \mathscr{H})}{\partial(X, Y)}. \tag{2.31}$$

Thus, shallow-water f-plane semigeostrophic theory consists of the predictive equation (2.31) for the potential thickness and the diagnostic equation (2.30) from which we obtain $\mathscr{H}$ from h^*. In (X, Y) space the advecting velocity for h^* is given in terms of $\mathscr{H}$. In this way the ageostrophic motions remain implicit in the coordinate transformation.

The original derivation of semigeostrophic theory (Hoskins 1975) was based on approximations to the momentum equations (2.1) and (2.2). A novel approach based on a variational principle was later used by Salmon (1983, 1985, 1988a,b) and Shutts (1989). An elegant aspect of this variational method is that approximations are introduced into Hamilton's principle in such a way that symmetries, and hence corresponding conservation laws, are preserved. In a related development and as an extension of the work of Cullen et al. (1991), Roulstone and Norbury (1994) have proposed a Hamiltonian approach that leads to a symplectic algorithm (Sanz-Serna and Calvo 1994) for calculating solutions to the semigeostrophic equations. In contrast, the approach we have taken here uses the transformed shallow-water equations (2.21)–(2.29) as the basis of approximation. This has the advantage of removing some of the mystery concerning the origin of the geostrophic coordinates.

Several recent studies have helped clarify our understanding of the accuracy of the semigeostrophic equations in comparison to the more accurate nonlinear balance models mentioned in footnote 1. With primarily oceanographic applications in mind, Allen et al. (1990a,b) and Barth et al. (1990) have presented a thorough analysis of the accuracy of the semigeostrophic model and several other intermediate models. Their results illustrate that, when comparing intermediate models, there is no guarantee that models with better conservation properties will produce more accurate solutions. In two related atmospheric studies, Snyder et al. (1991) have pointed out the systematic differences between semigeostrophic and primitive equation simulations of baroclinic waves, while Whitaker (1993) has demonstrated the high accuracy of the nonlinear balance model for such baroclinic waves.

In passing we note that the semigeostrophic approximation is only one of many possible approximations to (2.21)–(2.29). In fact, higher-order approximations are a subject of important current interest (e.g., Allen 1991; McIntyre and Norton 1994; Salmon 1993, personal communication; Warn et al. 1994). Our results indicate that, although such higher-order approximations lead to more complicated invertibility relations, they can be formulated in such a way as to retain the same predictive equation for h^* and the same form (2.24)–(2.25) for the advecting velocity.

d. Comments on arbitrariness in the Clebsch representation

There is a certain arbitrariness to the system (2.21)–(2.29) in the sense that the representation (2.8)–(2.9) is not the only representation leading to (2.6). To see this, note that a minor rearrangement allows (2.8) and (2.9) to be written as

$$u = \frac{\partial \hat{\chi}}{\partial x} - f(Y - y)\frac{\partial X}{\partial x}, \tag{2.8$'$}$$

$$v = \frac{\partial \hat{\chi}}{\partial y} - f(Y - y)\frac{\partial X}{\partial y} + f(X - x), \tag{2.9$'$}$$

where $\hat{\chi} = \chi + \frac{1}{2}f(X - x)(Y - y)$. If $\hat{\chi}$ is redefined as a new χ, then we obtain a different representation that also satisfies (2.6). This new representation leads to a somewhat different transformed set, but the original shallow-water equations (2.1)–(2.3) can still be recovered from it. Similarly, (2.8) and (2.9) can also be written as

$$u = \frac{\partial \tilde{\chi}}{\partial x} + f(X - x)\frac{\partial Y}{\partial x} - f(Y - y), \tag{2.8$''$}$$

$$v = \frac{\partial \tilde{\chi}}{\partial y} + f(X - x)\frac{\partial Y}{\partial y}, \tag{2.9$''$}$$

where $\tilde{\chi} = \chi - \frac{1}{2}f(X - x)(Y - y)$. If $\tilde{\chi}$ is redefined as a new χ, then we obtain a third representation resulting in (2.6). A further discussion of this point can be found in Eckart (1960). We can regard (2.8)–(2.9) as the definitions of the vorticity coordinates (X, Y) in terms of u, v, and χ. Our choice of (2.8)–(2.9) was in part motivated by the desire to represent both directions on the f plane equally.

3. The quasi-static primitive equations in spherical coordinates

a. Vorticity coordinate transformation of the primitive equations

Let us now generalize the shallow-water f-plane results of the previous section to the fully stratified primitive equations on the sphere. Rather than using potential temperature as the vertical coordinate, we use specific entropy $s = c_p \ln(T/T_0) - R \ln(p/p_0)$, where the pressure p and the absolute temperature T are related to the density ρ by the ideal gas law $p = \rho RT$ and where the subscript 0 denotes a constant reference value. The use of specific entropy rather than potential temperature results in a somewhat simpler hydrostatic equation, the right-hand side of which is simply the temperature rather than the Exner function. Using latitude ϕ and longitude λ as horizontal coordinates, we can write the quasi-static primitive equations as

$$\frac{\partial u}{\partial t} + \eta \dot{s} - \zeta v + \frac{\partial}{a\cos\phi\,\partial\lambda}\left[M + \frac{1}{2}(u^2 + v^2)\right] = 0, \quad (3.1)$$

$$\frac{\partial v}{\partial t} - \xi \dot{s} + \zeta u + \frac{\partial}{a\,\partial\phi}\left[M + \frac{1}{2}(u^2 + v^2)\right] = 0, \quad (3.2)$$

$$\frac{\partial M}{\partial s} = T, \quad (3.3)$$

$$\frac{D\sigma}{Dt} + \sigma\left(\frac{\partial u}{a\cos\phi\,\partial\lambda} + \frac{\partial(v\cos\phi)}{a\cos\phi\,\partial\phi} + \frac{\partial\dot{s}}{\partial s}\right) = 0, \quad (3.4)$$

where (u, v) are the zonal and meridional components of the wind, $\sigma = -\partial p/\partial s$ is the pseudodensity in s space, $M = c_pT + gz$ the Montgomery potential,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{a\cos\phi\,\partial\lambda} + v\frac{\partial}{a\,\partial\phi} + \dot{s}\frac{\partial}{\partial s} \quad (3.5)$$

the total derivative, and

$$(\xi, \eta, \zeta) = \left(-\frac{\partial v}{\partial s}, \frac{\partial u}{\partial s}, 2\Omega\sin\phi + \frac{\partial v}{a\cos\phi\,\partial\lambda} - \frac{\partial(u\cos\phi)}{a\cos\phi\,\partial\phi}\right). \quad (3.6)$$

We will now switch from the coordinates (λ, ϕ, s, t) to the new coordinates $(\Lambda, \Phi, S, \mathscr{T})$, where $S = s$ and $\mathscr{T} = t$. The symbols S and $\mathscr{T}$ are introduced to distinguish derivatives at fixed (Λ, Φ) (i.e., $\partial/\partial S$ and $\partial/\partial\mathscr{T}$) from derivatives at fixed (λ, ϕ) (i.e., $\partial/\partial s$ and $\partial/\partial t$). We require the new coordinates to be vorticity coordinates in the sense that

$$(\xi, \eta, \zeta) = 2\Omega\sin\Phi\left(\frac{a\,\partial(\Lambda, \sin\Phi)}{\partial(\phi, s)}, \frac{a\,\partial(\Lambda, \sin\Phi)}{\cos\phi\,\partial(s, \lambda)}, \frac{\partial(\Lambda, \sin\Phi)}{\partial(\lambda, \sin\phi)}\right). \quad (3.7)$$

If we equate the third entries of (3.6) and (3.7), we obtain an expression that can be rearranged in several ways, one of which is

$$\frac{\partial}{\partial\lambda}\left[v - \frac{1}{2}\Omega a(\Lambda - \lambda)\frac{\partial(\sin^2\Phi + \sin^2\phi)}{\partial\phi} + \frac{1}{2}\Omega a(\sin^2\Phi - \sin^2\phi)\frac{\partial\Lambda}{\partial\phi}\right] - \frac{\partial}{\partial\phi}\left[u\cos\phi - \frac{1}{2}\Omega a(\Lambda - \lambda)\frac{\partial(\sin^2\Phi)}{\partial\lambda} + \frac{1}{2}\Omega a(\sin^2\Phi - \sin^2\phi)\frac{\partial(\Lambda + \lambda)}{\partial\lambda}\right] = 0. \quad (3.8)$$

Thus, the first term in brackets can be expressed as $\partial\chi/a\partial\phi$ and the second term in brackets by $\partial\chi/a\partial\lambda$, where $\chi(\lambda, \phi, s, t)$ is a scalar potential. This results in

$$u\cos\phi = \frac{\partial\chi}{a\,\partial\lambda} + \frac{1}{2}\Omega a(\Lambda - \lambda)\frac{\partial(\sin^2\Phi)}{\partial\lambda} - \frac{1}{2}\Omega a(\sin^2\Phi - \sin^2\phi)\frac{\partial(\Lambda + \lambda)}{\partial\lambda}, \quad (3.9)$$

$$v = \frac{\partial\chi}{a\,\partial\phi} + \frac{1}{2}\Omega a(\Lambda - \lambda)\frac{\partial(\sin^2\Phi + \sin^2\phi)}{\partial\phi} - \frac{1}{2}\Omega a(\sin^2\Phi - \sin^2\phi)\frac{\partial\Lambda}{\partial\phi}. \quad (3.10)$$

Using (3.9) and (3.10) we can show that the first two entries of (3.7) are satisfied if

$$0 = \frac{\partial\chi}{\partial s} + \frac{1}{2}\Omega a^2(\Lambda - \lambda)\frac{\partial(\sin^2\Phi)}{\partial s} - \frac{1}{2}\Omega a^2(\sin^2\Phi - \sin^2\phi)\frac{\partial\Lambda}{\partial s}. \quad (3.11)$$

Since it does not contribute to either of the first two entries in (3.7), the scalar potential χ should not be interpreted as the velocity potential in the usual streamfunction–velocity potential decomposition of the horizontal wind field.

Just as in the shallow-water case of the previous section, (3.9)–(3.11) can be regarded as a Clebsch representation of the velocity field. The uniqueness discussion of section 2d also applies to the stratified case given in (3.9)–(3.11). The zero on the left-hand side of (3.11) is due to the fact that in the quasi-static system only the horizontal components of velocity appear in the right-hand side of (3.6). In a more general nonhydrostatic argument (in the z coordinate) the vertical velocity would contribute to the first two vorticity components in (3.6) and would also appear on the left-hand side of (3.11). The zero on the left-hand side of (3.11) can then be understood in terms of the neglect of the contribution of vertical motion to the kinetic energy in the variational principle for the quasi-static equations.

We can interpret the spherical geostrophic coordinates used by Magnusdottir and Schubert (1991) as an approximate form of (3.9)–(3.10). Similar to the f-plane case in the previous section, we first approximate $u\cos\phi - \partial\chi/a\partial\lambda$ by $u_g\cos\Phi$ and $v - \partial\chi/a\partial\phi$ by v_g. Then, we assume that $\partial\Lambda/\partial\lambda \approx 1$, $\partial\Phi/\partial\lambda \approx 0$, $\partial\Lambda/\partial\phi \approx 0$, and $\partial\Phi/\partial\phi \approx 1$ and, further, for (3.9) and (3.10) we use the approximations $\sin\Phi + \sin\phi \approx 2\sin\Phi$ and $\sin\Phi\cos\Phi + \sin\phi\cos\phi \approx 2\sin\Phi\cos\Phi$. The end result is $\Lambda = \lambda + (v_g/a2\Omega\sin\Phi\cos\Phi)$ and $\sin\Phi = \sin\phi - (u_g\cos\Phi/a2\Omega\sin\Phi)$, which are the spherical geostrophic coordinates.

Another special case of (3.9) occurs when the flow is zonally symmetric (i.e., $\partial\chi/\partial\lambda = 0$, $\partial\Phi/\partial\lambda = 0$, and $\partial\Lambda/\partial\lambda = 1$). Then, (3.9) reduces to $\Omega a\sin^2\Phi = \Omega a\sin^2\phi - u\cos\phi$ or, equivalently, $\Omega a^2\cos^2\Phi = \Omega a^2\cos^2\phi + ua\cos\phi$, the right-hand side of which is the absolute angular momentum per unit mass. Thus, in the zonally symmetric case, Φ is an angular momentum coordinate and represents the latitude to which a parcel must be moved in order to change its zonal velocity to zero. This "potential latitude" coordinate has proved useful in studies of the ITCZ and the Hadley circulation (Hack et al. 1989; Schubert et al. 1991) and will be further discussed in section 3c.

To transform the original primitive equations (3.1)–(3.3) we now take $\partial/\partial t$ of (3.9)–(3.11) to obtain

$$\frac{\partial(u\cos\phi)}{\partial t} + \frac{\partial}{a\partial\lambda}\left[M + \frac{1}{2}(u^2+v^2)\right] = a2\Omega\sin\Phi\frac{\partial(\Lambda, \sin\Phi)}{\partial(t,\lambda)} + \frac{\partial\mathscr{M}}{a\partial\lambda}, \quad (3.12)$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{a\partial\phi}\left[M + \frac{1}{2}(u^2+v^2)\right] = a2\Omega\sin\Phi\frac{\partial(\Lambda, \sin\Phi)}{\partial(t,\phi)} + \frac{\partial\mathscr{M}}{a\partial\phi}, \quad (3.13)$$

$$\frac{\partial}{\partial s}\left[M + \frac{1}{2}(u^2+v^2)\right] = a^2 2\Omega\sin\Phi\frac{\partial(\Lambda, \sin\Phi)}{\partial(t,s)} + \frac{\partial\mathscr{M}}{\partial s}, \quad (3.14)$$

where

$$\mathscr{M} = M + \frac{1}{2}(u^2+v^2) + \frac{\partial\chi}{\partial t} + \frac{1}{2}\Omega a^2(\Lambda - \lambda)\frac{\partial(\sin^2\Phi)}{\partial t} - \frac{1}{2}\Omega a^2(\sin^2\Phi - \sin^2\phi)\frac{\partial\Lambda}{\partial t}. \quad (3.15)$$

Adding $(\eta\dot{s} - \zeta v)\cos\phi$ to both sides of (3.12), $-\xi\dot{s} + \zeta u$ to both sides of (3.13), $\xi v - \eta u$ to both sides of (3.14), then using (3.6) and the original momentum equations (3.1)–(3.3), we obtain

$$2\Omega\sin\Phi\left(\frac{\partial\Phi}{\partial\lambda}\frac{D\Lambda}{Dt} - \frac{\partial\Lambda}{\partial\lambda}\frac{D\Phi}{Dt}\right)a^2\cos\Phi + \frac{\partial\mathscr{M}}{\partial\lambda} = 0, \quad (3.16)$$

$$2\Omega\sin\Phi\left(\frac{\partial\Phi}{\partial\phi}\frac{D\Lambda}{Dt} - \frac{\partial\Lambda}{\partial\phi}\frac{D\Phi}{Dt}\right)a^2\cos\Phi + \frac{\partial\mathscr{M}}{\partial\phi} = 0, \quad (3.17)$$

$$2\Omega\sin\Phi\left(\frac{\partial\Phi}{\partial s}\frac{D\Lambda}{Dt} - \frac{\partial\Lambda}{\partial s}\frac{D\Phi}{Dt}\right)a^2\cos\Phi + \frac{\partial\mathscr{M}}{\partial s} = T. \quad (3.18)$$

Together (3.16)–(3.18) imply that

$$(2\Omega V\sin\Phi, -2\Omega U\sin\Phi, T) = \left(\frac{\partial\mathscr{M}}{a\cos\Phi\partial\Lambda}, \frac{\partial\mathscr{M}}{a\partial\Phi}, \frac{\partial\mathscr{M}}{\partial S}\right), \quad (3.19)$$

where $U = a\cos\Phi D\Lambda/Dt$ and $V = aD\Phi/Dt$. The first entry in (3.19) has been obtained by eliminating $D\Lambda/Dt$ between (3.16) and (3.17), the second entry by eliminating $D\Phi/Dt$ between (3.16) and (3.17), and the third entry by substituting the first two into (3.18). Thus, (3.9)–(3.11) constitute the Clebsch representation of the velocity field while (3.19) constitutes the canonical quasi-static equations.

We can easily show that (3.5) can also be written in $(\Lambda, \Phi, S, \mathscr{T})$ space as

$$\frac{D}{Dt} = \frac{\partial}{\partial\mathscr{T}} + U\frac{\partial}{a\cos\Phi\partial\Lambda} + V\frac{\partial}{a\partial\Phi} + \dot{S}\frac{\partial}{\partial S}, \quad (3.20)$$

where $\dot{S} = \dot{s}$ since $S = s$. The advantage of (3.20) over (3.5) is that the horizontal advecting velocity is expressed in terms of derivatives of $\mathscr{M}$ by the first two entries of (3.19), which are mathematically analogous to the geostrophic formulas.

As in the shallow-water case, we have some freedom in the choice of $\mathscr{M}$, although the choice $\mathscr{M} = 0$ is not available because of the last entry in (3.19). Suppose we choose $\mathscr{M} = M + \frac{1}{2}(u^2+v^2)$. Then, if we multiply (3.1) by u and (3.2) by v, we obtain $D\mathscr{M}/Dt = \partial M/\partial t + \dot{s}\partial M/\partial s$. Applying (3.20) to $\mathscr{M}$ and using the first two entries of (3.19), we obtain $D\mathscr{M}/Dt = \partial\mathscr{M}/\partial\mathscr{T} + \dot{S}\partial\mathscr{M}/\partial S$. Since $\partial\mathscr{M}/\partial S = \partial M/\partial s$, subtraction of the previous two relations yields $\partial\mathscr{M}/\partial\mathscr{T} = \partial M/\partial t$.

The governing equation for the isentropic absolute vorticity can be derived from (3.1) and (3.2) or, equivalently, from the first two entries in (3.19). In either case it takes the form

$$\frac{D\zeta}{Dt} + \zeta\left(\frac{\partial u}{a\cos\phi\partial\lambda} + \frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi}\right) = \left(\xi\frac{\partial}{a\cos\phi\partial\lambda} + \eta\frac{\partial}{a\partial\phi}\right)\dot{s}. \quad (3.21)$$

Eliminating the isentropic divergence between (3.4) and (3.21) we obtain

$$\frac{DP}{Dt} = \frac{1}{\sigma}\left(\xi \frac{\partial}{a\cos\phi\partial\lambda} + \eta \frac{\partial}{a\partial\phi} + \zeta \frac{\partial}{\partial s}\right)\dot{s} = P\frac{\partial\dot{S}}{\partial S}, \tag{3.22}$$

where $P = \zeta/\sigma$ is the Rossby–Ertel potential vorticity. For adiabatic conditions, the potential vorticity P is conserved on fluid particles.

Now consider the potential pseudodensity, defined by

$$\sigma^* = \left(\frac{2\Omega\sin\Phi}{\zeta}\right)\sigma. \tag{3.23}$$

The potential vorticity and the potential pseudodensity are related by $P\sigma^* = 2\Omega\sin\Phi$. The potential pseudodensity equation, which can be easily obtained from the potential vorticity equation, takes the form

$$\frac{D\sigma^*}{Dt} + \sigma^*\left(\frac{\partial U}{a\cos\Phi\partial\Lambda} + \frac{\partial(V\cos\Phi)}{a\cos\Phi\partial\Phi} + \frac{\partial\dot{S}}{\partial S}\right) = 0, \tag{3.24}$$

which has a striking formal similarity to (3.4). In contrast to P, note that even in the absence of $\partial\dot{S}/\partial S$, σ^* is not conserved on fluid particles. The flux form of (3.24) is particularly convenient. With D/Dt given by (3.20), the flux form is given by

$$\frac{\partial\sigma^*}{\partial\mathscr{T}} + \frac{\partial(\sigma^* U)}{a\cos\Phi\partial\Lambda} + \frac{\partial(\sigma^* V\cos\Phi)}{a\cos\Phi\partial\Phi} + \frac{\partial(\sigma^*\dot{S})}{\partial S} = 0. \tag{3.25}$$

This is identical to the form of the potential pseudodensity equation found in many of the balanced model studies mentioned in the introduction. Here it has been derived from the quasi-static primitive equations on the sphere. An advantage of (3.25) is that the velocity that accomplishes the flux is simply expressed in terms of derivatives of $\mathscr{M}$ by (3.19).

The potential pseudodensity defined by (3.23) can also be expressed in terms of a Jacobian. To obtain this Jacobian form we first note that from the definition of pseudodensity we can write

$$\begin{aligned}\sigma &= -\frac{\partial p}{\partial s} = -\frac{\partial(\lambda, \sin\phi, p)}{\partial(\lambda, \sin\phi, s)} \\ &= -\frac{\partial(\Lambda, \sin\Phi, S)}{\partial(\lambda, \sin\phi, s)}\frac{\partial(\lambda, \sin\phi, p)}{\partial(\Lambda, \sin\Phi, S)} \\ &= -\left(\frac{\zeta}{2\Omega\sin\Phi}\right)\frac{\partial(\lambda, \sin\phi, p)}{\partial(\Lambda, \sin\Phi, S)}.\end{aligned} \tag{3.26}$$

Then, comparing (3.26) with (3.23), we obtain the Jacobian form

$$\frac{\partial(\lambda, \sin\phi, p)}{\partial(\Lambda, \sin\Phi, S)} + \sigma^* = 0. \tag{3.27}$$

This ends our discussion of the vorticity coordinate transformation of the primitive equations. An obvious direction for future work is the derivation of the vorticity coordinate transformations of generalized balanced models where, unlike the primitive equation model, only the σ^* field need be predicted. The foregoing primitive equation analysis can serve as a guide for such balanced model analysis. In the following subsection we outline the general structure of such balanced theories.

b. General structure for balanced theories

Balanced models are often obtained by approximating the original primitive equations (3.1)–(3.4) or, in Salmon's and Shutts's work, by approximating Hamilton's principle. The vorticity coordinate transformation of the primitive equations allows us to obtain balanced models through a different approach. This approach involves leaving the canonical equations (3.19) unchanged but approximating the Clebsch representations (3.9)–(3.10). An attractive feature of Salmon's and Shutts's work is that conservation relations are assured for classes of approximations that maintain certain symmetries in Hamilton's principle—one example being the association of particle labeling with potential vorticity (PV) conservation. The analogue in the vorticity coordinate transformation is that as long as the canonical equations (3.19) are not distorted, the isentropic absolute vorticity equation (3.21) is unchanged. Then, as long as the mass continuity equation is not distorted, the potential pseudodensity and potential vorticity principles are unchanged. This means a wide variety of approximations to (3.9) and (3.10) can be tried while PV conservation is guaranteed. An example of such a procedure (zonally symmetric balanced flow) is discussed below in section 3c. The general structure of such balanced theories can be seen as follows. First, write (3.27) in the form

$$\begin{vmatrix} \dfrac{\partial\lambda}{\cos\Phi\partial\Lambda} & \dfrac{\partial\sin\phi}{\cos\Phi\partial\Lambda} & \dfrac{\partial p}{\cos\Phi\partial\Lambda} \\ \dfrac{\partial\lambda}{\partial\Phi} & \dfrac{\partial\sin\phi}{\partial\Phi} & \dfrac{\partial p}{\partial\Phi} \\ \dfrac{\partial\lambda}{\partial S} & \dfrac{\partial\sin\phi}{\partial S} & \dfrac{\partial p}{\partial S} \end{vmatrix} + \sigma^* = 0. \tag{3.28}$$

Using the form of the hydrostatic equation given in the last entry of (3.19), we can write

$$p = p_0 e^{-S/R}\left(\frac{\partial\mathscr{M}}{T_0\partial S}\right)^{1/\kappa} \tag{3.29}$$

so that the third column of the determinant in (3.28) can be written entirely in terms of $\mathscr{M}$. If approximations to (3.9) and (3.10), along with a balance assumption, allow λ and $\sin\phi$ to be expressed in terms of $\mathscr{M}$, then (3.28) becomes an invertibility relation relating σ^* and

$\mathcal{M}$. This invertibility relation and the predictive equation (3.25) then give a succinct mathematical description of the dynamics. Thus, the pair of equations (3.25) and (3.28) constitute what might be called the canonical form for balanced models. The hemispherical semigeostrophic theory of Magnusdottir and Schubert (1991) has this canonical form.

c. Zonally symmetric balanced flow

As an example of the approximation procedure discussed in section 3b, let us consider zonally symmetric balanced flow. Under the zonal symmetry assumption, $\partial\chi/\partial\lambda = \partial\Phi/\partial\lambda = 0$ and $\partial\Lambda/\partial\lambda = 1$. Under the balance assumption, the left-hand side of (3.10) is neglected. Equations (3.9)–(3.11) then become

$$u\cos\phi = \Omega a(\sin^2\phi - \sin^2\Phi), \qquad (3.30)$$

$$0 = \frac{\partial\chi}{a\partial\phi} + \frac{1}{2}\Omega a(\Lambda - \lambda)\frac{\partial(\sin^2\Phi + \sin^2\phi)}{\partial\phi} - \frac{1}{2}\Omega a(\sin^2\Phi - \sin^2\phi)\frac{\partial\Lambda}{\partial\phi}, \qquad (3.31)$$

$$0 = \frac{\partial\chi}{\partial s} + \frac{1}{2}\Omega a^2(\Lambda - \lambda)\frac{\partial(\sin^2\Phi)}{\partial s} - \frac{1}{2}\Omega a^2(\sin^2\Phi - \sin^2\phi)\frac{\partial\Lambda}{\partial s}, \qquad (3.32)$$

while (3.15) reduces to

$$\mathcal{M} = M + \frac{1}{2}u^2 + \frac{\partial\chi}{\partial t} + \frac{1}{2}\Omega a^2(\Lambda - \lambda)\frac{\partial(\sin^2\Phi)}{\partial t} - \frac{1}{2}\Omega a^2(\sin^2\Phi - \sin^2\phi)\frac{\partial\Lambda}{\partial t}. \qquad (3.33)$$

Let us now take $\partial/\partial t$ of (3.30)–(3.32), and then proceed in a fashion analogous to that used in obtaining (3.12)–(3.14). This results in

$$\frac{\partial(u\cos\phi)}{\partial t} + \dot{s}\frac{\partial(u\cos\phi)}{\partial s} - \zeta v\cos\phi = 2\Omega\sin\Phi\left(-\frac{D\Phi}{Dt}\right)a\cos\Phi, \qquad (3.34)$$

$$\zeta u + \frac{\partial}{a\partial\phi}\left[M + \frac{1}{2}u^2\right] = 2\Omega\sin\Phi\left(\frac{\partial\Phi}{\partial\phi}\frac{D\Lambda}{Dt} - \frac{\partial\Lambda}{\partial\phi}\frac{D\Phi}{Dt}\right)a\cos\Phi + \frac{\partial\mathcal{M}}{a\partial\phi}, \qquad (3.35)$$

$$\frac{\partial M}{\partial s} = 2\Omega\sin\Phi\left(\frac{\partial\Phi}{\partial s}\frac{D\Lambda}{Dt} - \frac{\partial\Lambda}{\partial s}\frac{D\Phi}{Dt}\right)a^2\cos\Phi + \frac{\partial\mathcal{M}}{\partial s}. \qquad (3.36)$$

Since we are leaving the canonical equations (3.19) unchanged, (3.16)–(3.18) also remain unchanged and, under zonal symmetry, the right-hand sides of (3.34) and (3.35) vanish while the right-hand side of (3.36) equals T, yielding

$$\frac{Du}{Dt} - \left(2\Omega\sin\phi + \frac{u\tan\phi}{a}\right)v = 0, \qquad (3.37)$$

$$\left(2\Omega\sin\phi + \frac{u\tan\phi}{a}\right)u + \frac{\partial M}{a\partial\phi} = 0, \qquad (3.38)$$

$$\frac{\partial M}{\partial s} = T, \qquad (3.39)$$

which are the approximate momentum equations for the zonal balanced model. Note that another way of writing the right-hand side of (3.34) is $D/Dt(\Omega a\cos^2\Phi)$ or equivalently, $D/Dt(\Omega a\cos^2\phi + u\cos\phi)$. Thus, the vanishing of the right-hand side of (3.34) is simply a consequence of the conservation of absolute angular momentum.

To express the entire problem in terms of a predictive equation for σ^* and a diagnostic invertibility principle, we first note that the zonal symmetry assumption reduces (3.25) to

$$\frac{\partial\sigma^*}{\partial\mathcal{T}} + \frac{\partial(\sigma^*\dot{S})}{\partial S} = 0, \qquad (3.40)$$

while (3.28) reduces to

$$\begin{vmatrix} \dfrac{\partial\sin\phi}{\cos\Phi\partial\Phi} & \dfrac{\partial p}{\cos\Phi\partial\Phi} \\ \dfrac{\partial\sin\phi}{\partial S} & \dfrac{\partial p}{\partial S} \end{vmatrix} + \sigma^* = 0. \qquad (3.41)$$

If we choose $\mathcal{M} = M + \frac{1}{2}u^2$, the zonal balance relation (3.38) can be expressed as

$$2\Omega^2 a\sin\Phi\cos\Phi\left(\frac{\sin^2\Phi - \sin^2\phi}{1 - \sin^2\phi}\right) = \frac{\partial\mathcal{M}}{a\partial\Phi}, \qquad (3.42)$$

which can be solved for $\sin\phi$ in terms of $\mathcal{M}$. When the $\sin\phi$ determined from (3.42) and the p determined from (3.29) are substituted into (3.41), we obtain a second-order nonlinear partial differential equation relating $\mathcal{M}$ to σ^*. With appropriate boundary conditions, we can regard this second-order equation as the invertibility relation. Alternatively, we can regard (3.41), (3.42), and (3.29) as a system of three diagnostic equations for the three unknowns $\mathcal{M}$, $\sin\phi$, and p with given σ^*.

Solutions of this zonally symmetric balanced model for intertropical convergence zone (ITCZ) heat sources have been presented by Schubert et al. (1991). For typical ITCZ heating patterns the solutions of (3.40) show the development of a strip of high PV at lower levels and low PV at upper levels. After only a few days of ITCZ convection, significant reversals in the northward gradient of PV are produced on the poleward side of the ITCZ at low levels and on the equatorial side at upper levels. This sets the stage for ITCZ breakdown through combined barotropic–baroclinic instability. It is interesting to note that, although the meridional motion is not explicit in the system (3.40), (3.41), (3.42), and (3.29), it can easily be recovered from solutions of the invertibility principle at consecutive times. This can be understood by noting that part of the output of the invertibility solver is the function $\phi(\Phi, S)$, which can be regarded as giving meridional particle positions.

4. Concluding remarks

The purpose of this paper has been to formulate vorticity coordinates and to derive the potential pseudodensity equation in order to illuminate a dynamical structure shared by the primitive equations and a class of balanced models. All of these models automatically conserve potential vorticity. One would also expect that they all possess wave activity conservation laws such as the primitive equations (Haynes 1988) and the semigeostrophic equations (Kushner and Shepherd 1994). The potential pseudodensity equation is closely related to the potential vorticity equation, but it has certain advantages. One advantage of potential pseudodensity over potential vorticity involves the behavior of the two fields across the equator. Since potential vorticity is generally positive in the Northern Hemisphere, negative in the Southern Hemisphere, and vanishes near the equator, the isolines of potential vorticity in a meridional cross section are nearly vertical near the equator. This means that potential vorticity, which is so useful in distinguishing tropospheric and stratospheric air in midlatitudes, loses this usefulness in the Tropics. Potential pseudodensity is generally nonnegative, and its isolines in a meridional cross section are nearly horizontal across the equator. It can be used to distinguish tropospheric and stratospheric air globally. A second advantage of potential pseudodensity over potential vorticity involves the treatment of isentropic surfaces intersecting the ground using the massless layer approach. This, of course, is intimately related to the dynamics of baroclinic waves and fronts. In the massless layer, potential vorticity is infinite but potential pseudodensity is zero. This makes potential pseudodensity a more natural variable in calculations with balanced models that use the massless layer approach.

Of course, there are many applications in which potential vorticity is a more convenient variable than potential pseudodensity. For example, it is more desirable to work with potential vorticity when considering wave–mean flow interactions and conditions of stability of a basic-state flow and when we are confined to working in physical coordinates, vorticity coordinates not being available. The former point is related to the fact that for the large-scale flow we are generally considering Rossby wave interactions, but the restoring mechanism for Rossby waves involves the gradient of potential vorticity, the largest component of which is often the contribution due to the earth's rotation. The second point arises because potential pseudodensity is defined in terms of the latitudinal part of the vorticity coordinates so that it is essential to have those coordinates available.

We have here used the transformed primitive equations in two ways—as a basis for balanced approximations and to illuminate a simple structure for all balanced models. However, the transformed primitive equations and their associated vorticity coordinates have potential uses beyond balanced model applications. For example, geostrophic coordinates have recently been explored for use in objective analysis of fronts (Desroziers and Lafore 1993). The vorticity coordinates used in the present paper also hold promise for such applications. In fact, they may be regarded as the natural coordinates to consider for primitive equation analysis under highly anisotropic conditions since they share the "stretching property" of geostrophic coordinates and they are more general. Several advantages of the transformed primitive equations have already been mentioned: for example, the way in which the potential pseudodensity equation arises naturally as one of the prognostic equations and the way in which the canonical momentum equations allow the total time derivative to be written in simple form where the advection is expressed solely in terms of derivatives of a potential. Note that in the fully spherical system, a set of vorticity coordinates is essential to the definition of potential pseudodensity; that is, $\sin\Phi$ appears in the definition (3.23). Also, even for adiabatic and frictionless motion, the potential pseudodensity is not necessarily conserved following a fluid parcel. These two facts distinguish potential pseudodensity from its close relative, the potential vorticity. The most intriguing aspect of the transformed set is its connection to balanced models. We have seen how semigeostrophic theory can be obtained from its shallow-water equation equivalent and how the zonally symmetric balanced model can be obtained from the fully stratified transformed equations. In fact, the transformed set can serve as the basis of a whole ensemble of balanced models, for example, three-dimensional semigeostrophic models and zonally symmetric or axisymmetric two-dimensional gradient balanced models. An important topic for future research is the formulation of a three-dimensional balanced model of this class whose balance conditions clearly go beyond geostrophic balance.

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