NOTES AND CORRESPONDENCE

Semigeostrophic Theory

WAYNE H. SCHUBERT

Department of Atmospheric Science, Colorado State University, Fort Collins, CO 80523

20 June 1984 and 4 April 1985

ABSTRACT

The geopotential tendency form of semigeostrophic theory is derived and compared with the potential vorticity form. The tendency form is compact and particularly convenient for non-Boussinesq, nonuniform potential vorticity flows.

1. Introduction

By combining the geostrophic momentum approximation (Eliassen, 1948) and the geostrophic coordinate transformation, Hoskins (1975) and Hoskins and Draghici (1977) have derived a filtered, three dimensional system of equations (semigeostrophic equations) which are nearly as simple as the quasi-geostrophic equations but which apply to more general physical situations such as fronts, jets, occluding baroclinic waves, etc. With a specified, height independent deformation field, two dimensional versions of the semigeostrophic equations (Hoskins, 1971; Hoskins and Bretherton, 1972) give fairly realistic simulations of both surface fronts (uniform potential vorticity model) and upper-level fronts (discontinuous potential vorticity model). In the three dimensional case uniform potential vorticity jet flows develop unstable baroclinic waves which evolve into the nonlinear regime, producing fronts and an occluding warm sector (Hoskins, 1976; Hoskins and West, 1979; Hoskins and Heckley, 1980). Three dimensional nonuniform potential vorticity jet flows produce similar evolution except that upper tropospheric frontogenesis is more realistic (Heckley and Hoskins, 1982).

In this note we derive the "geopotential tendency" form of semigeostrophic theory. This form is convenient because the ageostrophic circulation remains implicit and only one elliptic equation needs to be solved each time step. From the computational point of view integrations of the nonuniform potential vorticity case become almost as easy as integrations of the uniform potential vorticity case.

2. Review of semigeostrophic theory

a. Geostrophic momentum approximation

Our starting point is the f-plane system of equations with the geostrophic momentum approximation [see Eqs. (1)–(5) of Table 1]. Here we use the pseudo-height $z = [1 - (p/p_0)^\kappa]c_p\theta_0/g$ as the vertical coordinate. The pseudo-density $\rho = \rho_0[1 - gz/c_p\theta_0]^{(1-\kappa)/\kappa}$ is then a known function of z. In (1)–(5), $(u_g, v_g) = f^{-1}(-\partial\phi/\partial y, \partial\phi/\partial x)$ are the geostrophic components of the wind, (u_a, v_a) the ageostrophic components, and $(u, v) = (u_g + u_a, v_g + v_a)$ the components of the total wind.

Although the derivation is lengthy, it can be shown that the above system has the three-dimensional vorticity equation

$$\frac{D\zeta}{Dt} = (\zeta \cdot \nabla)\mathbf{u} + \zeta w \frac{\partial \rho}{\rho \partial z} - \frac{g}{\theta_0}\mathbf{k} \times \nabla\theta, \tag{6}$$

where $\mathbf{u} = (u, v, w)$, $D/Dt = \partial/\partial t + u(\partial/\partial x) + v(\partial/\partial y) + w(\partial/\partial z)$, and

$$\zeta = \left(-\frac{\partial v_g}{\partial z} + \frac{1}{f}\frac{\partial(u_g, v_g)}{\partial(y, z)}, \frac{\partial u_g}{\partial z} + \frac{1}{f}\frac{\partial(u_g, v_g)}{\partial(z, x)}, f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f}\frac{\partial(u_g, v_g)}{\partial(x, y)}\right). \tag{7}$$

Combining (6) with the continuity equation (4) and the thermodynamic equation (5) we obtain the potential vorticity equation

$$\frac{Dq}{Dt} = 0, \tag{8}$$

where

$$q = \frac{g}{\rho\theta_0 f}\zeta \cdot \nabla\theta. \tag{9}$$

b. Semigeostrophic equations

Introducing the geostrophic coordinates

$$(X, Y, Z, T) = \left(x + \frac{v_g}{f}, y - \frac{u_g}{f}, z, t\right) \tag{10}$$

and the potential function

$$\Phi = \phi + \frac{1}{2}(u_g^2 + v_g^2), \tag{11}$$

Hoskins (1975) has shown that the geostrophic and hydrostatic relations take the form

TABLE 1. Semigeostrophic theory: Equations.

Geostrophic momentum approximation			Semigeostrophic equations	
$\frac{\partial u_g}{\partial t} + u\frac{\partial u_g}{\partial x} + v\frac{\partial u_g}{\partial y} + w\frac{\partial u_g}{\partial z} - fv_a = 0$	(1)		$\frac{\partial u_g}{\partial T} + u_g\frac{\partial u_g}{\partial X} + v_g\frac{\partial u_g}{\partial Y} - fv_a^* = 0$	(15)
$\frac{\partial v_g}{\partial t} + u\frac{\partial v_g}{\partial x} + v\frac{\partial v_g}{\partial y} + w\frac{\partial v_g}{\partial z} + fu_a = 0$	(2)		$\frac{\partial v_g}{\partial T} + u_g\frac{\partial v_g}{\partial X} + v_g\frac{\partial v_g}{\partial Y} + fu_a^* = 0$	(16)
$\frac{\partial \phi}{\partial z} = \frac{g}{\theta_0}\theta$	(3)	Geostrophic coordinates ⟹	$\frac{\partial \Phi}{\partial Z} = \frac{g}{\theta_0}\theta$	(17)
$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \rho w}{\rho \partial z} = 0$	(4)		$\frac{\partial u_a^*}{\partial X} + \frac{\partial v_a^*}{\partial Y} + \frac{\partial \rho w^*}{\rho \partial Z} = 0$	(18)
$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} = 0$	(5)		$\frac{\partial \theta}{\partial T} + u_g\frac{\partial \theta}{\partial X} + v_g\frac{\partial \theta}{\partial Y} + \frac{\theta_0}{g}q\rho w^* = 0$	(19)

$$\left(fv_g, -fu_g, \frac{g}{\theta_0}\theta\right) = \left(\frac{\partial \Phi}{\partial X}, \frac{\partial \Phi}{\partial Y}, \frac{\partial \Phi}{\partial Z}\right), \tag{12}$$

while the potential vorticity becomes

$$q = \frac{g}{\rho\theta_0}\frac{\zeta}{f}\frac{\partial \theta}{\partial Z}, \tag{13}$$

where ζ is the vertical component of absolute vorticity. Since $D/DT = \partial/\partial T + u_g(\partial/\partial X) + v_g(\partial/\partial Y) + w(\partial/\partial Z)$, the horizontal advection is now geostrophic. Then, introducing the transformed ageostrophic components

$$(u_a^*, v_a^*, w^*) = \left(u_a + \frac{w}{f}\frac{\partial v_g}{\partial z}, v_a - \frac{w}{f}\frac{\partial u_g}{\partial z}, \frac{f}{\zeta}w\right), \tag{14}$$

Hoskins and Draghici (1977) showed that (1)–(5) can be written[1] as (15)–(19) of Table 1.

Although the system (15)–(19) can be regarded as closed, it is not convenient for calculation. The ageostrophic components (u_a^*, v_a^*, w^*) are implicit and must be determined in such a way that the predictions of (u_g, v_g) from (15)–(16) and θ from (19) are consistent with a continued state of geostrophic and hydrostatic balance. In Sections 3 and 4 we discuss two alternate forms of the semigeostrophic equations.

3. Potential vorticity form

From (15) and (16) we can obtain

$$D_g\left(f\mathbf{k} \times \frac{\partial \mathbf{v}_g}{\partial Z}\right) - f^2\frac{\partial \mathbf{v}_a^*}{\partial Z} = \frac{1}{2}\mathbf{Q}, \tag{20}$$

where $D_g = \partial/\partial T + u_g(\partial/\partial X) + v_g(\partial/\partial Y)$ and $\mathbf{Q} = -2g/\theta_0(\partial \mathbf{v}_g/\partial X \cdot \nabla\theta, \partial \mathbf{v}_g/\partial Y \cdot \nabla\theta)$, with ∇ denoting the horizontal del operator in geostrophic space. From (19) we can also obtain

$$D_g\left(\frac{g}{\theta_0}\nabla\theta\right) + \nabla(\rho q w^*) = \frac{1}{2}\mathbf{Q}. \tag{21}$$

Since the thermal wind equation [derived from (12)] can be written

$$f\mathbf{k} \times \frac{\partial \mathbf{v}_g}{\partial Z} + \frac{g}{\theta_0}\nabla\theta = 0, \tag{22}$$

(20) and (21) can be added to yield

$$\nabla(\rho q w^*) - f^2\frac{\partial \mathbf{v}_a^*}{\partial Z} = \mathbf{Q}. \tag{23}$$

Taking the horizontal divergence of (23) and combining the result with the continuity equation (18) we obtain (24), the w^* equation given at the top of the left column in Table 2.

Using (12)–(14) the conservation of potential vorticity can be written as (25). Finally, the relation between q and Φ, and the thermodynamic equation applied at the boundaries can be written as (26). Equations (24)–(26) constitute a closed set in the unknowns Φ, q, w^*. This is the set discussed by Hoskins (1982, p. 145) and used by Heckley and Hoskins (1982). Numerical integration proceeds in the order given in Table 2. Note that two elliptic equations (one for w^*, one for Φ) must be solved each time step.

There is a special case which results in considerable simplification of (24)–(26). This is the case of uniform potential vorticity flow. In this case w^* need not be computed from (24), and q in (26) is a constant. Thus, the whole dynamics reduces to (26). The Boussinesq approximation (ρ = constant) leads to even further simplification. A great deal can be learned about fronts and baroclinic waves using this Boussinesq, uniform potential vorticity model (e.g., Hoskins, 1976; Hoskins and West, 1979).

4. Geopotential tendency form

We now derive the form of semigeostrophic theory which involves Φ_T. Since $\Phi_T = \phi_t$ we call this the geopotential tendency form.

[1] Hoskins and Draghici (1977) use the Boussinesq form of (18), but their analysis easily generalizes to the non-Boussinesq case given in Table 1.

TABLE 2. Potential vorticity and geopotential tendency forms of semigeostrophic theory.

Potential vorticity form		Geopotential tendency form	
$\left\{\begin{array}{l}\nabla^2(\rho q w^*) + f^2 \dfrac{\partial}{\partial Z}\left(\dfrac{\partial \rho w^*}{\rho \partial Z}\right) = \nabla \cdot \mathbf{Q} \\ \text{B.C. } w^* = 0 \quad \text{at} \quad Z = 0, Z_T\end{array}\right\}$	(24)		
$\left(\dfrac{\partial}{\partial T} - \dfrac{1}{f}\dfrac{\partial \Phi}{\partial Y}\dfrac{\partial}{\partial X} + \dfrac{1}{f}\dfrac{\partial \Phi}{\partial X}\dfrac{\partial}{\partial Y} + \dfrac{q}{\Phi_{ZZ}} w^* \dfrac{\partial}{\partial Z}\right) q = 0$	(25)	$\rho q = \dfrac{f^2 \Phi_{ZZ}}{f^2 - (\Phi_{XX} + \Phi_{YY})}$	(29)
$\left\{\begin{array}{l}\dfrac{1}{f^2}\nabla^2\Phi + \dfrac{1}{\rho q}\dfrac{\partial^2 \Phi}{\partial Z^2} = 1 \\ \text{B.C.}\left(\dfrac{\partial}{\partial T} - \dfrac{1}{f}\dfrac{\partial \Phi}{\partial Y}\dfrac{\partial}{\partial X} + \dfrac{1}{f}\dfrac{\partial \Phi}{\partial X}\dfrac{\partial}{\partial Y}\right)\dfrac{\partial \Phi}{\partial Z} = 0 \quad \text{at} \quad Z = 0, Z_T\end{array}\right\}$	(26)	$\left\{\begin{array}{l}\rho \nabla^2 \Phi_T + \dfrac{\partial}{\partial Z}\left(\dfrac{f^2}{q}\dfrac{\partial \Phi_T}{\partial Z}\right) = \nabla \cdot \mathbf{P} \\ \text{B.C.} \dfrac{f^2}{q}\dfrac{\partial \Phi_T}{\partial Z} = \mathbf{k} \cdot \mathbf{P} \quad \text{at} \quad Z = 0, Z_T\end{array}\right\}$	(30)

Adding $\partial[\rho f \cdot (16)]/\partial X$, $\partial[-\rho f \cdot (15)]/\partial Y$ and $\partial[(f^2/q)(g/\theta_0) \cdot (19)]/\partial Z$, then using (12) and (18) we obtain

$$\frac{\partial}{\partial X}\left[\rho D_g\left(\frac{\partial \Phi}{\partial X}\right)\right] + \frac{\partial}{\partial Y}\left[\rho D_g\left(\frac{\partial \Phi}{\partial Y}\right)\right] + \frac{\partial}{\partial Z}\left[\frac{f^2}{q} D_g\left(\frac{\partial \Phi}{\partial Z}\right)\right] = 0. \quad (27)$$

Defining

$$\mathbf{P} = \left(\rho \frac{\partial \mathbf{v}_g}{\partial X} \cdot \nabla \Phi, \rho \frac{\partial \mathbf{v}_g}{\partial Y} \cdot \nabla \Phi, \frac{f^2}{q}\frac{\partial \mathbf{v}_g}{\partial Z} \cdot \nabla \Phi\right), \quad (28)$$

(27) can be written as Eq. (30) of Table 2. Equation (26) can now be rearranged to the form given by (29). Equations (29) and (30) constitute a closed set in Φ_T and q. Only one elliptic equation needs to be solved each time step. The ageostrophic circulation remains entirely implicit. If w^* is desired, it can be easily computed from (19) using the known Φ_T. In the special case of uniform potential vorticity flow it can be shown that $\nabla \cdot \mathbf{P} = 0$, so that (26) and (30) become essentially equivalent.

Equation (30) is somewhat analogous to an equation derived by Eliassen (1974) in a θ-coordinate type model with the geostrophic momentum approximation. The relation between (30) and the potential vorticity equation (25) has been discussed in the context of Eliassen's balanced vortex model (Schubert and Hack, 1983; Schubert *et al.*, 1983) by Thorpe (1985).

5. Concluding remarks

The main result obtained here is the right-hand side of Table 2. This geopotential tendency form of semigeostrophic theory is compact and allows numerical integration of non-Boussinesq, nonuniform potential vorticity flows to be implemented almost as simply as the Boussinesq, uniform potential vorticity case. The word "almost" must be used because for the Boussinesq, uniform potential vorticity case the interior part of (26) can be solved analytically and numerical integration need be implemented only on the boundary condition part of (26), as described by Hoskins (1976); in contrast, the non-Boussinesq, nonuniform potential vorticity case requires that we numerically solve the interior part of (30). Although repeatedly solving three dimensional elliptic problems can be time consuming, fast multigrid solvers are becoming available. Finally we note that the distinction between the potential vorticity form and the geopotential tendency form disappears when the additional quasi-geostrophic approximations are introduced into (1)–(5).

Acknowledgments. The author is grateful to S. Fulton, M. DeMaria, P. Ciesielski and H.-C. Kuo for their valuable comments, and to O. Panella for her help in preparing the manuscript. This research was supported by NSF Grant ATM-8207563.

REFERENCES

Eliassen, A., 1948: The quasi-static equations of motion. *Geofys. Publ.*, **17**(3).

——, 1974: On the semi-geostrophic approximation with application to barotropic-baroclinic instability. *Lectures in Colloquium on Subsynoptic Extratropical Weather Systems: Observations, Analysis, Modeling and Prediction, Vol. I*, NCAR, Boulder, CO.

Heckley, W. A., and B. J. Hoskins, 1982: Baroclinic waves and frontogenesis in a non-uniform potential vorticity semi-geostrophic model. *J. Atmos. Sci.*, **39**, 1999–2016.

Hoskins, B. J., 1971: Atmospheric frontogenesis: some solutions. *Quart. J. Roy. Meteor. Soc.*, **97**, 139–153.

——, 1975: The geostrophic momentum approximation and the semi-geostrophic equations. *J. Atmos. Sci.*, **32**, 233–242.

——, 1976: Baroclinic waves and frontogenesis. Part I: Introduction and Eady waves. *Quart. J. Roy. Meteor. Soc.*, **102**, 103–122.

——, and F. P. Bretherton, 1972: Atmospheric frontogenesis models: Mathematical formulation and solution. *J. Atmos. Sci.*, **29**, 11–37.

——, and I. Draghici, 1977: The forcing of ageostrophic motion according to the semi-geostrophic equations and in an isentropic coordinate model. *J. Atmos. Sci.*, **34**, 1859–1867.

——, and N. V. West, 1979: Baroclinic waves and frontogenesis. Part II: Uniform potential vorticity jet flows—cold and warm fronts. *J. Atmos. Sci.*, **36**, 1663–1680.

——, and W. A. Heckley, 1980: Cold and warm fronts in baroclinic waves. *Quart. J. Roy. Meteor. Soc.*, **107**, 79–90.

Schubert, W. H., and J. J. Hack, 1983: Transformed Eliassen balanced vortex model. *J. Atmos. Sci.*, **40**, 1571–1583.

——, P. E. Ciesielski, and J. J. Hack, 1983: Response of the tropical atmosphere to convective cloud clusters. *Proc. the ECMWF Workshop on Convection in Large-Scale Models*, Reading, 377–390.

Thorpe, A. J., 1985: Diagnosis of balanced vortex structure using potential vorticity. *J. Atmos. Sci.*, **42**, 397–406.