The Generalization of Semigeostrophic Theory to the β-Plane

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ABSTRACT

We develop here the isentropic–geostrophic coordinate version of semigeostrophic theory on a midlatitude β-plane. This approach results in a simple mathematical form in which the horizontal ageostrophic velocities are implicit and the entire dynamics reduces to a predictive equation for the potential pseudodensity and an invertibility relation. Linearized versions of the theory lead to a generalized Charney–Stern theorem for combined barotropic–baroclinic instability and to Rossby wave solutions with a meridional structure different from that in quasi-geostrophic theory.

1. Introduction

By combining the geostrophic momentum approximation and the geostrophic coordinate transformation, Hoskins (1975) and Hoskins and Draghici (1977) have derived a filtered, three-dimensional system of semigeostrophic equations that are nearly as simple as the quasi-geostrophic equations but that apply to more general physical situations such as fronts, jets, and occluding baroclinic waves. One limitation of this form of semigeostrophic theory is that it does not include a variable Coriolis parameter. Recently, Salmon (1983, 1985, 1988) has shown that Hamiltonian methods can be used to generate dynamical approximations with useful conservation properties and that these methods also suggest transformations to new variables in which the physics takes a simple mathematical form. As an application of this approach, Salmon (1985) and Shutts (1989) have investigated the inclusion of a variable Coriolis parameter in semigeostrophic theory. In particular, Shutts has derived an elegant planetary semigeostrophic theory, with perhaps the only major assumption being the neglect of the kinetic energy associated with the component of velocity parallel to the axis of rotation. The goal of the present paper is to examine this question of a variable Coriolis parameter in semigeostrophic theory, not through the use of Hamiltonian methods, but by more conventional and elementary methods of analysis. In particular, we present semigeostrophic theory on a midlatitude β-plane in what we believe is the most elegant and concise version—that version which makes simultaneous use of isentropic and geostrophic coordinates. With these coordinates, semigeostrophic theory reduces to two equations—a predictive equation for the potential pseudodensity and a diagnostic equation (or invertibility principle) whose solution yields the balanced wind and mass fields from the potential pseudodensity. In isentropic and geostrophic coordinates the divergent part of the circulation remains entirely implicit.

The outline of our approach is as follows. In section 2 we present a generalization of the geostrophic momentum approximation to the midlatitude β-plane [Eqs. (2.1)–(2.7)] and show that this β-plane geostrophic momentum approximation satisfies an energy principle just like the primitive equations except that the kinetic energy is evaluated geostrophically. We then derive the potential pseudodensity equation (section 3) and the invertibility relation (section 4), the combination of which constitutes the entire dynamics. In sections 5 and 6 we examine some consequences of linearized versions of this theory, in particular the form taken by the Charney–Stern theorem on combined barotropic–baroclinic instability and the meridional structure of Rossby waves.

2. Semigeostrophic equations on the β-plane

Let us approximate the variable Coriolis parameter as a linear function of the north–south coordinate. As approximations to the isentropic coordinate version of the primitive equations on the β-plane let us consider

$$\frac{Du_g}{Dt} - (f(Y)v + \beta(y - Y)v_g) + \frac{\partial M}{\partial x} = 0, \quad (2.1)$$

$$\frac{Dv_g}{Dt} + (f(Y)u + \beta(y - Y)u_g) + \frac{\partial M}{\partial y} = 0, \quad (2.2)$$

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$$\frac{\partial M}{\partial \theta} = \Pi, \tag{2.3}$$

$$\frac{D\sigma}{Dt} + \sigma\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \dot{\theta}}{\partial \theta}\right) = 0, \tag{2.4}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \dot{\theta}\frac{\partial}{\partial \theta} \tag{2.5}$$

is the total derivative, $M = \theta\Pi + gz$ the Montgomery potential, $\Pi = c_p(p/p_0)^\kappa$ the Exner function, $\sigma = -\partial p/\partial\theta$ the pseudodensity, (u, v) the eastward and northward components of the velocity, (u_g, v_g) the geostrophic wind components given by

$$(v_g, -u_g) = \frac{1}{f(Y)}\left(\frac{\partial M}{\partial x}, \frac{\partial M}{\partial y}\right), \tag{2.6}$$

and Y the northward coordinate in Salmon's generalized geostrophic coordinates

$$(X, Y) = \left(x + \frac{v_g}{f(Y)}, y - \frac{u_g}{f(Y)}\right). \tag{2.7}$$

The above set of equations might be called the β-plane geostrophic momentum approximation. We postpone (until section 7) discussion of the motivation for the approximate forms (2.1) and (2.2), but simply note that these two equations revert to the primitive equations if u_g and v_g are replaced by u and v, and to the f-plane geostrophic momentum approximation when $\beta = 0$ (Eliassen 1948; Hoskins 1975). The β terms in (2.1) and (2.2) can be regarded as corrections for the fact that f is taken at Y rather than y. In (2.1), (2.2), (2.6) and (2.7) we have explicitly indicated the dependence of f on Y. Henceforth, we shall use a more compact notation so that whenever f appears in the following discussion it should be interpreted as $f(Y)$.

The kinetic energy equation associated with (2.1)–(2.4) is

$$\frac{\partial}{\partial t}(\sigma K_g) + \frac{\partial}{\partial x}(\sigma u(K_g + \phi)) + \frac{\partial}{\partial y}(\sigma v(K_g + \phi)) + \frac{\partial}{\partial \theta}\left(\sigma\dot{\theta}(K_g + \phi) - \phi\frac{\partial p}{\partial t}\right) + \sigma\alpha\omega = 0, \tag{2.8}$$

where $K_g = \frac{1}{2}(u_g^2 + v_g^2)$ is the geostrophic kinetic energy. Multiplying (2.4) by c_pT we obtain the thermodynamic energy equation

$$\frac{\partial}{\partial t}(\sigma c_pT) + \frac{\partial}{\partial x}(\sigma u c_pT) + \frac{\partial}{\partial y}(\sigma v c_pT) + \frac{\partial}{\partial \theta}(\sigma\dot{\theta}c_pT) - \sigma\alpha\omega = \sigma Q. \tag{2.9}$$

Adding (2.8) and (2.9) we obtain the total energy equation

$$\frac{\partial}{\partial t}(\sigma(K_g + c_pT)) + \frac{\partial}{\partial x}(\sigma u(K_g + M)) + \frac{\partial}{\partial y}(\sigma v(K_g + M)) + \frac{\partial}{\partial \theta}\left(\sigma\dot{\theta}(K_g + M) - \phi\frac{\partial p}{\partial t}\right) = \sigma Q. \tag{2.10a}$$

Before integrating (2.10a), we adopt an idea that has proved useful in such contexts as the definition of available potential energy (Lorenz 1955), the analysis of baroclinic instability (Bretherton 1966; Hoskins et al. 1985; Hsu 1988), and the finite amplitude Eliassen–Palm theorem (Andrews 1983). The idea involves what happens when an isentropic surface intersects the earth's surface. We can regard such an isentrope as continuing just under the earth's surface with a pressure equal to the surface pressure. At any horizontal position where two distinct isentropic surfaces run just under the earth's surface (and hence have the same pressure), there is no mass trapped between them, so that $\sigma = 0$ there. Let us regard the bottom isentropic surface as the largest value of θ that remains everywhere below the earth's surface. Assuming the top boundary is both an isentropic and isobaric surface, assuming no topography and vanishing $\dot{\theta}$ at the top and bottom, we can integrate (2.10a) over the entire atmosphere to obtain

$$\frac{\partial}{\partial t}\iiint (K_g + c_pT)\sigma dxdyd\theta = \iiint Q\sigma dxdyd\theta. \tag{2.10b}$$

Thus, except for the fact that the kinetic energy is evaluated geostrophically, the governing equations (2.1)–(2.7) have a total energy conservation principle identical to the one that exists for the primitive equations.

Here we are concerned with the simultaneous use of isentropic and generalized geostrophic coordinates because this will lead to an elegant version of the potential pseudodensity equation. Defining $\Theta = \theta$ and $T = t$ (but noting that $\partial/\partial\theta$ and $\partial/\partial t$ imply fixed x, y while $\partial/\partial\Theta$ and $\partial/\partial T$ imply fixed X, Y), we can show that derivatives in (x, y, θ, t) space are related to derivatives in (X, Y, Θ, T) space by

$$\frac{\partial}{\partial t} = \frac{\partial X}{\partial t}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial t}\frac{\partial}{\partial Y} + \frac{\partial}{\partial T}, \tag{2.11}$$

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y}, \tag{2.12}$$

$$\frac{\partial}{\partial y} = \frac{\partial X}{\partial y}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial y}\frac{\partial}{\partial Y}, \tag{2.13}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial X}{\partial \theta}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial \theta}\frac{\partial}{\partial Y} + \frac{\partial}{\partial \Theta}. \tag{2.14}$$

Applying (2.11)–(2.14) to the Bernoulli function $M^* = M + \frac{1}{2}(u_g^2 + v_g^2)$, it can be shown that

$$\left(\frac{\partial M}{\partial x}, \frac{\partial M}{\partial y}, \frac{\partial M}{\partial \theta}, \frac{\partial M}{\partial t}\right) = \left(\frac{\partial M^*}{\partial X}, \frac{\partial M^*}{\partial Y} - \frac{\beta}{f}(u_g^2 + v_g^2), \frac{\partial M^*}{\partial \Theta}, \frac{\partial M^*}{\partial T}\right). \quad (2.15)$$

The additional term $\beta(u_g^2 + v_g^2)/f$ in the meridional transformation generally constitutes a small correction. For midlatitude flows the ratio of the magnitude of this term to the magnitude of the $\partial M^*/\partial y$ term is the ratio of the characteristic magnitude of horizontal wind to the product of the Coriolis parameter and the earth's radius, which is typically about 1 to 60. For mathematical consistency we shall retain this term, although in section 4 we shall point out some simplifications that result from its neglect.

The transformation relations (2.11)–(2.14) also imply that the total derivative (2.5) can be written as

$$\frac{D}{Dt} = \frac{\partial}{\partial T} + \frac{DX}{Dt}\frac{\partial}{\partial X} + \frac{DY}{Dt}\frac{\partial}{\partial Y} + \dot{\theta}\frac{\partial}{\partial \Theta}. \quad (2.16)$$

With the aid of (2.15) we can now easily show that (2.1) and (2.2) take the form

$$f\frac{DY}{Dt} = \frac{\partial M^*}{\partial X}, \quad (2.17)$$

$$-f\frac{DX}{Dt} = \frac{\partial M^*}{\partial Y}. \quad (2.18)$$

When (2.17) and (2.18) are used on (2.16), we see that a major advantage of the transformation from (x, y, θ, t) space to (X, Y, Θ, T) space is the absence of ageostrophic advection in (2.16). We shall take advantage of this in section 3.

3. Vorticity, potential vorticity, and potential pseudodensity equations

The equation for the absolute *isentropic* vorticity can be derived from (2.1) and (2.2) or from (2.17) and (2.18); the latter approach is simpler (in this regard see also the approach to potential vorticity conservation suggested by Shutts and Cullen (1987), section 2). Combining the derivatives of (2.17) and (2.18) in such a way as to form the total derivative of $f\partial(X, Y)/\partial(x, y)$, i.e., forming $X_x(2.17)_y - Y_y(2.18)_x - X_y(2.17)_x + Y_x(2.18)_y$, results in

$$\frac{D\zeta}{Dt} + \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)\dot{\theta} = 0, \quad (3.1)$$

where

$$(\xi, \eta, \zeta) = f\left(\frac{\partial(X, Y)}{\partial(y, \theta)}, \frac{\partial(X, Y)}{\partial(\theta, x)}, \frac{\partial(X, Y)}{\partial(x, y)}\right). \quad (3.2)$$

Eliminating the horizontal divergence between (2.4) and (3.1) we obtain

$$\sigma\frac{DP}{Dt} = \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial\dot{\theta}}{\partial\theta}\right)\dot{\theta} = \zeta\frac{\partial\dot{\theta}}{\partial\Theta}, \quad (3.3)$$

where $P = \zeta/\sigma$ is the Rossby–Ertel potential vorticity. The last step in (3.3) follows from (3.2) and (2.12)–(2.14). This shows that $\partial/\partial\Theta$ is actually the derivative along the vorticity vector and thus the name "vortex coordinates" might be appropriate for (X, Y, Θ, T).

Let us now define the potential pseudodensity σ^* by

$$\sigma^* = \frac{f}{\zeta}\sigma. \quad (3.4)$$

Note that the potential vorticity P and the potential pseudodensity σ^* are related by $P\sigma^* = f$. Thus, when (3.3) is manipulated into an equation for σ^* and when the total derivative is expressed in geostrophic space using (2.16), we can write the fundamental predictive equation of the model in the flux form

$$\frac{\partial\sigma^*}{\partial T} + \frac{\partial}{\partial X}\left(-\frac{\sigma^*}{f}\frac{\partial M^*}{\partial Y}\right) + \frac{\partial}{\partial Y}\left(\frac{\sigma^*}{f}\frac{\partial M^*}{\partial X}\right) + \frac{\partial}{\partial\Theta}(\sigma^*\dot{\theta}) = 0. \quad (3.5)$$

4. Invertibility principle

The potential pseudodensity σ^* is a combination of the mass field σ and the wind field $\partial(x, y)/\partial(X, Y)$. However, since σ is related to M^* through hydrostatic balance and $\partial(x, y)/\partial(X, Y)$ is related to M^* through geostrophic balance, σ^* depends only on M^*. Thus, everything can be obtained from σ^* if we can somehow invert it to obtain M^*. The relation between M^* and σ^* is derived as follows. From the definition of σ^* we have

$$\frac{\partial(x, y, \Pi)}{\partial(X, Y, \Theta)} + \Gamma\sigma^* = 0, \quad (4.1)$$

where $\Gamma = d\Pi/dp = \kappa\Pi/p$. Using the geostrophic, hydrostatic, and coordinate transformation relations, we can express x, y and Π in terms of M^* as

$$(x, y, \Pi) = [X - f^{-2}M_X^*, Y - \hat{f}^{-2}(M_Y^* - \beta f^{-3}M_X^{*2}), M_\Theta^*], \quad (4.2)$$

where we have used the shorthand notation $\hat{f}^2 = f(Y)(2f(Y) - f(y))$. Substituting (4.2) into (4.1), we obtain

$$\frac{1}{f^2\hat{f}^2}\begin{vmatrix} M^*_{XX} - f^2 & \hat{f}^2[\hat{f}^{-2}(M^*_Y - \beta f^{-3}M^{*2}_X)]_X & M^*_{\Theta X} \\ f^2(f^{-2}M^*_X)_Y & \hat{f}^2[\hat{f}^{-2}(M^*_Y - \beta f^{-3}M^{*2}_X)]_Y - \hat{f}^2 & M^*_{\Theta Y} \\ M^*_{X\Theta} & \hat{f}^2[\hat{f}^{-2}(M^*_Y - \beta f^{-3}M^{*2}_X)]_\Theta & M^*_{\Theta\Theta} \end{vmatrix} + \Gamma\sigma^* = 0, \quad (4.3a)$$

which expresses the invertibility principle in terms of the determinant of a Hessian-type matrix. Shutts and Cullen (1987) have discussed in detail the relation of hydrodynamic stability and the positive definiteness of such matrices. Since we are neglecting the effects of topography and assuming that the lower boundary is the constant height surface $z = 0$ and the isentropic surface $\Theta = \Theta_B$, then $M = \Theta\Pi$ at $\Theta = \Theta_B$. Written in terms of M^*, this lower boundary condition becomes

$$\Theta M^*_\Theta - M^* + \frac{1}{2f^2}[M^{*2}_X + (M^*_Y + 2\beta f^{-1} \times (\Theta M^*_\Theta - M^*))^2] = 0 \quad \text{at} \quad \Theta = \Theta_B. \quad (4.3b)$$

Since the upper boundary is an isentropic and isobaric surface with potential temperature Θ_T and pressure p_T, the upper boundary condition for (4.3a) is simply

$$M^*_\Theta = \Pi(p_T) \quad \text{at} \quad \Theta = \Theta_T. \quad (4.3c)$$

The lateral boundary conditions depend on the particular application, but typically might consist of a zonally periodic midlatitude channel. In any event, for a given σ^*, we can regard (4.3) as a nonlinear second order problem in M^*. Although Γ and $\hat{f}$ both depend on M^*, this additional nonlinearity is weak. In fact, if the additional term in the second entry of (2.15) is neglected, the invertibility relation (4.3a) simplifies to

$$\frac{1}{f^4}\begin{vmatrix} M^*_{XX} - f^2 & f^2(f^{-2}M^*_X)_Y & M^*_{\Theta X} \\ M^*_{XY} & f^2(f^{-2}M^*_Y)_Y - f^2 & M^*_{\Theta Y} \\ M^*_{X\Theta} & M^*_{Y\Theta} & M^*_{\Theta\Theta} \end{vmatrix} + \Gamma\sigma^* = 0, \quad (4.4a)$$

and the lower boundary condition to

$$\Theta M^*_\Theta - M^* + \frac{1}{2f^2}(M^{*2}_X + M^{*2}_Y) = 0 \quad \text{at} \quad \Theta = \Theta_B. \quad (4.4b)$$

The upper boundary condition (4.3c) is unchanged. In (4.4) the nonlinearity associated with $\hat{f}$ has disappeared.

The mathematical problems (4.3) or (4.4) are generalizations of the f-plane case discussed by Schubert et al. (1989). In particular, when f is assumed to be a constant, (4.3) and (4.4) reduce to their (2.14), while if we further assume that $\partial/\partial Y = 0$, the middle elements of the first and third rows and the first and third columns vanish, in which case (4.3) and (4.4) reduce to their (3.2). An efficient multigrid solver for the two-dimensional f-plane case has been developed by Fulton (1989).

The predictive equation (3.5) and invertibility principle (4.3) or (4.4) form a closed system for the prediction of σ^* and the diagnosis of M^*. Since the problem of isentropes intersecting the earth's surface has been addressed by adopting the massless region approach outlined in section 2, the system can in principle handle surface frontogenesis. Since $\sigma = 0$ in the massless region, $\sigma^* = 0$ there also. Thus, the prediction of σ^* by (3.5) includes predicting the movement of the $\sigma^* = 0$ region. This procedure is consistent with Bretherton's (1966) notion that "any flow with potential temperature variations over a horizontal rigid plane boundary may be considered equivalent to a flow without such variations, but with a concentration of potential vorticity very close to the boundary." We have simply replaced Bretherton's thin sheet of infinite potential vorticity with a thin sheet of zero potential pseudodensity and chosen to predict the evolution of the entire σ^* field (including the zero potential pseudodensity region) with (3.5). Of course, such a procedure has implications for the numerical methods used since we must cope with discontinuities in σ^*. However, workable schemes do exist. For example, recently Arakawa and Hsu (see chapter V of Hsu 1988), in the context of solving (2.4) in a primitive equation model, have proposed a finite difference scheme that has very small dissipation and computational dispersion and that guarantees positive definiteness.

5. The generalized Charney–Stern theorem

Since a major application of semigeostrophic theory is the study of baroclinic wave processes, it is of interest to derive the form of the Charney–Stern (1962) theorem that results from the present β-plane semigeostrophic theory. Here we follow the approach of Eliassen (1983), which avoids the assumption of exponential time dependence. We begin by linearizing the potential pseudodensity equation about a zonal flow that varies in Y and Θ, to obtain

$$D_T\sigma^{*\prime} + v'_g f(f^{-1}\bar{\sigma}^*)_Y = 0, \quad (5.1)$$

where $D_T = \partial/\partial T - f^{-1}(\partial\bar{M}^*/\partial Y)(\partial/\partial X)$, and $fv'_g = M^{*\prime}_X$. Introducing the northward geostrophic particle displacement η', defined by $v'_g = D_T\eta'$, we can integrate (5.1) to obtain

$$\sigma^{*\prime} + f(f^{-1}\bar{\sigma}^*)_Y\eta' = 0. \quad (5.2)$$

Multiplying (5.2) by fv'_g and taking the zonal average at fixed Y, we obtain

$$\left(f^2(f^{-1}\bar{\sigma}^*)_Y \frac{1}{2}\overline{\eta'^2}\right)_T + f\overline{v'_g\sigma^{*\prime}} = 0. \quad (5.3)$$

The linearized invertibility relation can be written as

$$\sigma^{*\prime} = \bar{\sigma}^* x'_X + \bar{y}_\Theta p'_Y - \bar{y}_Y p'_\Theta + \bar{p}_Y y'_\Theta - \bar{p}_\Theta y'_Y, \quad (5.4)$$

where $x' = -v'_g/f$ and $y' = u'_g/f$. Multiplying (5.4) by fv'_g and taking the zonal average, we find that

$$\begin{aligned} f\overline{v'_g\sigma^{*\prime}} &= \overline{fv'_g(\bar{y}_\Theta p'_Y - \bar{y}_Y p'_\Theta)} + \overline{fv'_g(\bar{p}_Y y'_\Theta - \bar{p}_\Theta y'_Y)} \\ &= (f\bar{y}_\Theta\overline{v'_g p'} - f\bar{p}_\Theta\overline{v'_g y'})_Y + (f\bar{p}_Y\overline{v'_g y'} - f\bar{y}_Y\overline{v'_g p'})_\Theta, \end{aligned} \quad (5.5)$$

where the last step requires use of the zonally averaged thermal wind equation. The second line of (5.5) shows that the northward geostrophic flux of potential pseudodensity can be expressed as the divergence of a geostrophic Eliassen–Palm flux. When (5.5) is substituted into (5.3) we obtain an equation for the density of E–P wave activity (see Edmon et al. 1980, for a discussion of the quasi-geostrophic version). We are now interested in integrating this equation over the (Y, Θ) plane. If we use the boundary conditions $v'_g = 0$ at the northern and southern boundaries (Eliassen 1983), the resulting lateral boundary fluxes vanish. The resulting boundary flux at the model top vanishes because both $\bar{p}_Y$ and p' vanish there. To show that the lower boundary flux vanishes, we proceed as follows. From the lower boundary condition on the basic state flow we have $\Theta\bar{\Pi} - M = 0$. Differentiating this with respect to Y and using the geostrophic relation, we obtain

$$\Theta\bar{\Gamma}\bar{p}_Y + f\bar{u}_g\bar{y}_Y = 0 \quad \text{at} \quad \Theta = \Theta_B. \quad (5.6)$$

From the lower boundary condition on the perturbation flow we have $\Theta\Pi' - M^{*\prime} + \bar{u}_g u'_g = 0$. Multiplying this by v'_g and taking the zonal average, we obtain

$$\Theta\bar{\Gamma}\overline{v'_g p'} + f\bar{u}_g\overline{v'_g y'} = 0 \quad \text{at} \quad \Theta = \Theta_B. \quad (5.7)$$

Subtracting $\bar{y}_Y$ times (5.7) from $\overline{v'_g y'}$ times (5.6), we conclude that

$$\bar{p}_Y\overline{v'_g y'} - \bar{y}_Y\overline{v'_g p'} = 0 \quad \text{at} \quad \Theta = \Theta_B. \quad (5.8)$$

Thus, the integration of (5.3) over Y and Θ results in

$$\frac{\partial}{\partial T}\int\int f^2(f^{-1}\bar{\sigma}^*)_Y \frac{1}{2}\overline{\eta'^2}\,dY d\Theta = 0. \quad (5.9)$$

This is essentially Eliassen's (1983) generalization of the Charney–Stern theorem, which states that, in order for $\overline{\eta'^2}$ to grow, $(f^{-1}\bar{\sigma}^*)_Y$ must have both signs.

6. Rossby wave solutions

It is of interest to examine the Rossby wave solutions within semigeostrophic theory since they have a different meridional structure than those obtained from midlatitude β-plane quasi-geostrophic theory. For simplicity let us consider the quasi-Boussinesq case in which Γ is set equal to the constant $\Gamma_0 = R/p_B$. Then, for a resting basic state with constant pseudodensity $\sigma_0 = (p_B - p_T)/(\Theta_T - \Theta_B)$, the linearized potential pseudodensity equation (5.1) and the linearized invertibility relation (5.4) can be combined into

$$\frac{\partial}{\partial T}\left\{\frac{\partial^2 M^{*\prime}}{\partial X^2} + f^2\frac{\partial}{\partial Y}\left(\frac{1}{f^2}\frac{\partial M^{*\prime}}{\partial Y}\right) + \frac{f^2}{\Gamma_0\sigma_0}\frac{\partial^2 M^{*\prime}}{\partial\Theta^2}\right\} + \beta\frac{\partial M^{*\prime}}{\partial X} = 0. \quad (6.1)$$

Defining $c^2 = \Gamma_0\sigma_0(\Theta_T - \Theta_B)^2 = R(\Theta_T - \Theta_B)(p_B - p_T)/p_B$ and $\mathcal{Y} = (\beta c)^{-1/2}(f_0 + \beta Y)$, and then searching for solutions of the form

$$M^{*\prime}(X, \mathcal{Y}, \Theta, T) = \mathcal{M}(\mathcal{Y})\cos\left[\frac{\alpha_m(\Theta_T - \Theta)}{(\Theta_T - \Theta_B)}\right]e^{i(kX+\nu T)}, \quad (6.2)$$

we obtain the meridional structure equation

$$\frac{d^2\mathcal{M}}{d\mathcal{Y}^2} - \frac{2}{\mathcal{Y}}\frac{d\mathcal{M}}{d\mathcal{Y}} + \left(\frac{ck}{\nu} - \frac{ck^2}{\beta} - \alpha_m{}^2\mathcal{Y}^2\right)\mathcal{M} = 0. \quad (6.3)$$

The linearized lower boundary condition is satisfied if the constants α_m are the solutions of the transcendental equation $\alpha_m \tan\alpha_m = (\Theta_T - \Theta_B)/\Theta_B$. For the US Standard Atmosphere we have $\Theta_T = 333$ K, $p_T = 22.5$ kPa, $\Theta_B = 287$ K, $p_B = 100$ kPa, in which case $c = 101.2$ m s^{-1} and the solutions of the transcendental equation are $\alpha_m \approx 0.1241\pi$, 1.0160π, 2.0081π, 3.0054π, $\cdots$, corresponding to $m = 0, 1, 2, 3, \cdots$. The first of these roots ($m = 0$) corresponds to the external mode, while the remaining roots (which are approximately integral multiples of π) correspond to internal modes.

The solution of (6.3) is (Abramowitz and Stegun 1964, page 505)

$$\mathcal{M}(\mathcal{Y}) = A\left[U\left(a, \frac{5}{2}, \alpha_m\mathcal{Y}_S^2\right)M\left(a, \frac{5}{2}, \alpha_m\mathcal{Y}^2\right) - M\left(a, \frac{5}{2}, \alpha_m\mathcal{Y}_S^2\right)U\left(a, \frac{5}{2}, \alpha_m\mathcal{Y}^2\right)\right]\mathcal{Y}^3 \times \exp\left(-\frac{1}{2}\alpha_m\mathcal{Y}^2\right), \quad (6.4)$$

where A is an arbitrary normalization constant, $M(a, 5/2, \alpha_m\mathcal{Y}^2)$ and $U(a, 5/2, \alpha_m\mathcal{Y}^2)$ are the confluent hypergeometric functions (Kummer functions), $\mathcal{Y}_S$ and $\mathcal{Y}_N$ are the southern and northern boundaries, and where the parameter a is related to k, α_m and ν by the dispersion relation

$$\nu = \frac{\beta k}{k^2 + \beta\alpha_m(5 - 4a)/c}. \quad (6.5)$$

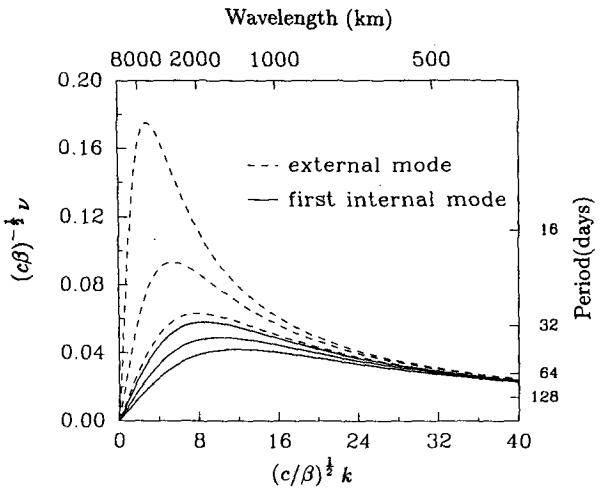

FIG. 1. The dimensionless Rossby wave frequency $(c\beta)^{-1/2}\nu$ as a function of the dimensionless zonal wavenumber $(c/\beta)^{1/2}k$ as computed from the dispersion relation (6.5). The three dashed curves are for the external modes (from top down) $a_{0,n}$, $n = 0, 1, 2$. Similarly, the three solid curves are for the first internal modes $a_{1,n}$, $n = 0, 1, 2$.

The meridional wavenumber parameter a must be determined from specification of the lateral boundary conditions, which are here assumed to be walls (with $v_g' = 0$ and hence $\mathcal{M} = 0$) at both southern and northern boundaries. Equation (6.4) already satisfies the boundary condition $\mathcal{M} = 0$ at $\mathcal{Y} = \mathcal{Y}_S$. Requiring this condition also at $\mathcal{Y} = \mathcal{Y}_N$ leads to the transcendental relation

$$U\left(a, \frac{5}{2}, \alpha_m \mathcal{Y}_S^2\right) M\left(a, \frac{5}{2}, \alpha_m \mathcal{Y}_N^2\right) - M\left(a, \frac{5}{2}, \alpha_m \mathcal{Y}_S^2\right) U\left(a, \frac{5}{2}, \alpha_m \mathcal{Y}_N^2\right) = 0. \quad (6.6)$$

For fixed values of α_m, $\mathcal{Y}_S$ and $\mathcal{Y}_N$, this determines a set of a's, which we label $a_{m,n}$. For a 3000 km wide β-plane channel centered at 45°N we have $\mathcal{Y}_S = 1.947$ and $\mathcal{Y}_N = 3.147$. For the external mode ($\alpha_0 \approx 0.1241\pi$) the first three noninteger[1] solutions of (6.6) are $a_{0,n} \approx -3.9836, -17.1795, -39.1563$ while for the first internal mode ($\alpha_1 \approx 1.0160\pi$) they are $a_{1,n} \approx -4.1225, -6.2532, -8.9528$. The dispersion relation (6.5) and the corresponding structure function $\mathcal{M}(\mathcal{Y})$ are shown in Figs. 1 and 2. In standard quasi-geostrophic β-plane theory the three f^2 factors in (6.1) are replaced by the constant f_0^2, which results in a trigonometric variation of $M^{*\prime}$ in the meridional direction. Thus, the midlatitude quasi-geostrophic and semigeostrophic Rossby wave solutions essentially differ only in meridional structure. The deviation of the curves in Fig. 2 from pure sine waves reflects the influence of variable f in (6.1).

7. Concluding remarks

The main results obtained here are (3.5) and (4.3). Together these form a system for the prediction of the potential pseudodensity σ^* and the diagnosis of M^* from σ^*. The advantage of using geostrophic and isentropic coordinates is that the ageostrophic components of the flow have become entirely implicit in the coordinate transformation, a result stemming from the fact that (2.1), (2.2), (2.6) and (2.7) lead to the canonical momentum equations (2.17) and (2.18). The approximations (2.1) and (2.2) may at first sight seem subtle. However, they are less so if they are viewed simply as the equations that result from combining the *desired* canonical forms (2.17) and (2.18) with the geostrophic relations (2.6) and the geostrophic coordinates (2.7). The canonical forms (2.17) and (2.18) are *desired* because, according to (2.16), they make the ageostrophic advection implicit in the coordinate transformation. From this point of view we can regard the present model as one of a family, with the other members derived by retaining (2.17) and (2.18), modifying (2.6) and/or (2.7), and then finding the resulting versions of (2.1) and (2.2). In fact, this technique of beginning with the geostrophic relations, the coordinate transformation relations and the desired canonical momentum equations, and then deriving the

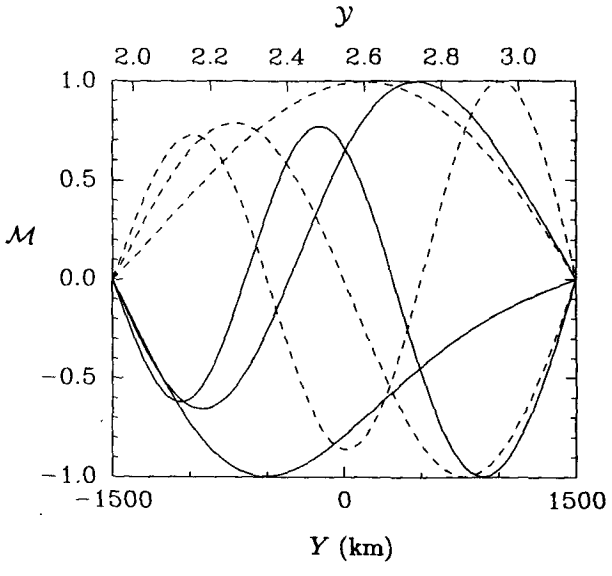

FIG. 2. The meridional structure functions $\mathcal{M}(\mathcal{Y})$ computed from (6.4). The normalization factor A in (6.4) has been chosen so that the maximum value of $|\mathcal{M}(\mathcal{Y})|$ is unity. The external modes ($a_{0,n}$; $n = 0, 1, 2$) are given by the dashed lines and the first internal modes ($a_{1,n}$; $n = 0, 1, 2$) by the solid lines. The index n gives the number of nodes in the interior of the domain.

[1] Negative integer solutions of (6.6) also exist but they result in trivial solutions for $\mathcal{M}(\mathcal{Y})$.

generalized geostrophic momentum approximation, can be used to extend the present β-plane argument to spherical coordinates. This will be discussed in a forthcoming paper.

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