Transport and mixing in idealized barotropic hurricane-like vortices

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ABSTRACT: The effective diffusivity diagnostic is used to obtain basic insight into the two-dimensional transport and mixing properties of idealized barotropic tropical-storm and hurricane-like vortices. Three flow configurations believed to be relevant to hurricane dynamics are examined in a non-divergent barotropic model: (i) an elliptical vortex, (ii) a Rankine vortex in a turbulent background vorticity field, and (iii) unstable vorticity rings. During the evolution of these vortical flows, effective diffusivity is used as a mixing diagnostic on a passive tracer field that also evolves in the non-divergent flow. The internal dynamical processes causing mixing, as well as the location and magnitude of both turbulent mixing and partial barrier regions, are identified in the evolving vortices. Breaking vortex Rossby waves (VRWs) are found to create turbulent mixing regions of finite radial extent. For monotonic vortices, which are analogous to tropical storms, the wave breaking and axisymmetrization creates a surf zone outside the radius of maximum wind, while the vortex core remains a partial barrier or containment vessel. For unstable vorticity rings, which are analogous to intensifying hurricanes, two regimes of internal mixing are found. During barotropic instability of thick rings, the inner and outer breaking VRWs create two local mixing regions, separated by a partial barrier region at the location of the tangential jet. For barotropic instability of thin rings, the entire hurricane inner core becomes a turbulent mixing region, allowing passive tracers to be radially mixed between the eye, eyewall and local environment. In either case, the horizontal mixing associated with the inner, breaking VRW would support intensification, provided the passive tracer is equivalent potential temperature with a maximum in the eye. In addition to the insights obtained for internal mixing in hurricanes, effective diffusivity is shown to be a robust diagnostic for two-dimensional turbulence, complementing its previous use in large-scale atmospheric dynamics. Copyright © 2009 Royal Meteorological Society

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1. Introduction

Although the intensification and decay of tropical cyclones is partially controlled by factors such as large-scale vertical wind shear and moist entropy flux from the underlying ocean, internal dynamical processes are also important and not clearly understood (for a succinct review of our current understanding, see Wang and Wu, 2004). Some important internal processes are wave-mean flow interaction due to convectively coupled vortex Rossby waves (Montgomery and Kallenbach, 1997; Möller and Montgomery, 1999), potential vorticity (PV) mixing between the eyewall and eye (Schubert et al., 1999; Kossin and Schubert, 2001; Montgomery et al., 2002), vortex Rossby-wave breaking and inner spiral rainbands (Guinn and Schubert, 1993; Chen and Yau, 2001; Wang 2008), and eyewall replacement cycles (Willoughby et al., 1982; Houze et al., 2007). Accurate predictions of hurricane intensity are currently limited by the lack of a comprehensive understanding of these internal processes. In particular, these internal processes may be important factors governing the rapid intensification and weakening of hurricanes. As an example of the potential importance of mixing processes in the hurricane inner core, Figure 1 shows the persistent multiple mesovortices that occurred in Hurricane Isabel (2003). It is generally agreed that such mesovortices can play an important role in the radial transport of PV in the hurricane core. A topic of recent research and debate (Persing and Montgomery, 2003; Aberson et al., 2006; Braun et al., 2006; Montgomery et al., 2006; Cram et al., 2007; Bryan and Rotunno, 2009) is whether such multiple mesovortices can transport high moist entropy from the eye to the eyewall, and thereby explain why both real and model hurricanes often have an intensity that is significantly higher than that predicted by the axisymmetric MPI theory of Emanuel (1986, 1988, 1995).

Mixing is due to the combined effect of differential advection and turbulent (or, inevitably, molecular) diffusion. Differential advection (i.e. stirring) stretches and deforms material lines, from which diffusion accomplishes true irreversible mixing. The interplay between advection and diffusion makes mixing difficult to quantify. Recent work has proposed the use of a hybrid Eulerian–Lagrangian area coordinate...
Eye mesovortices such as these can efficiently transport quasi-passive tracers such as cloud water between the eyewall and the eye. Figure 1. Visible satellite image of Hurricane Isabel at 1315 UTC on 12 September 2003 (reproduced from Kossin and Schubert, 2004). It is well known that geophysical vortices act as transport barriers. However, in local regions of the vortices and their near environment, strong mixing can occur. For example, it has been shown that planetary Rossby waves may break on wintertime stratospheric vortices, creating a nonlinear critical layer (or surf zone), by which filaments can be extruded from the edge of the vortex (McIntyre and Palmer, 1983, 1984). These long filamentary structures can then mix chemical species from the vortex to the midlatitudes (Waugh et al., 1994).

In hurricanes, horizontal mixing due to vortex Rossby wave activity is occurring at smaller scales, helping to determine the spatial distributions of both quasi-passive tracers (e.g. moist entropy or total airborne moisture) and active tracers (e.g. vorticity or potential vorticity).

Insight into transport and mixing processes in a full physics numerical model simulation of a hurricane undergoing vertical shear has been obtained by Cram et al. (2007). An important finding of that work was that high equivalent potential temperature ($\theta_e$) air was transported from the eye into the eyewall, thereby increasing the efficiency of the hurricane heat engine. In the present work, we seek to obtain a more basic understanding of transport and mixing processes in hurricanes by considering idealized vortices in a non-divergent barotropic model framework. While the non-divergent barotropic model is an oversimplification of the real atmosphere, it can capture low-frequency (Rossby wave) advective dynamics of tropical cyclone evolution in a dynamically clean framework. Numerical solutions to the non-divergent barotropic vorticity equation and the advection–diffusion equation are obtained with suitable initial conditions, and the effective diffusivity diagnostic is used to quantify the transport and mixing properties of the following tropical-storm and hurricane-like flows: (i) an elliptical vorticity field, (ii) a Rankine vortex embedded in a turbulent background vorticity field, and (iii) unstable vorticity rings. These configurations have been shown to exhibit some interesting internal dynamics relevant to tropical cyclone evolution, such as secondary eyewall formation, vortex Rossby-wave-breaking surf zones and PV mixing between the eye and eyewall. While the vorticity dynamics involved in these scenarios has been studied before, here we use the effective diffusivity diagnostic to quantify their mixing properties, leading to a better understanding of both passive tracer transport in hurricanes and the internal dynamics governing hurricane intensity change. The location and magnitude of strong partial barriers (where the time-scale for transport is large), weak partial barriers (where the time-scale for transport is small) and mixing regions (where trajectories are chaotic) are identified in these vortices. Implications for the evolution of passive tracers and their relationship to intensity change are discussed in light of the results.

The outline of this article is as follows. In section 2 the dynamical model and passive tracer equation used for this study are described. In section 3 we review the derivation of the transformation of the advection–diffusion equation into the area coordinate and the equivalent radius coordinate, yielding the effective diffusivity diagnostic in a form useful for hurricane studies. In section 4 we present pseudospectral model results for several types of mixing scenarios believed to be relevant in hurricane dynamics. In section 5 we document the relative insensitivity of the effective diffusivity diagnostic to certain arbitrary choices made in its calculation from solutions of the passive tracer equation. Finally, the main conclusions of this study are presented in section 6.
2. Dynamical model and passive tracer equation

The dynamical model used here considers two-dimensional, non-divergent motions on a plane. The governing vorticity equation is

\[
\frac{\partial \xi}{\partial t} + \mathbf{u} \cdot \nabla \xi = \nu \nabla^2 \xi, \tag{1}
\]

where \( \mathbf{u} = \mathbf{k} \times \nabla \psi \) is the horizontal, non-divergent velocity, \( \xi = \nabla^2 \psi \) is the relative vorticity and \( \nu \) is the constant viscosity. The solutions presented here were obtained with a double Fourier pseudospectral code having 768 \times 768 equally spaced points on a doubly periodic 600 km \times 600 km domain. Since the code was run with a de-aliased calculation of the nonlinear term in (1), there were 256 \times 256 resolved Fourier modes. The wavelength of the highest Fourier mode is 2.3 km. A fourth-order Runge–Kutta scheme was used for time-differencing, with a 3.5 s time step. The value of viscosity was chosen to be \( \nu = 50 \text{ m}^2 \text{s}^{-1} \), so the characteristic damping time for modes having total wavenumber equal to 256 is 2.4 h, while the damping time for modes having total wavenumber equal to 170 is 5.5 h.

As a way to understand the transport and mixing properties of an evolving flow described by (1), it is useful to also calculate the evolution of a passive tracer subject to diffusion and advection by the non-divergent velocity \( \mathbf{u} \). The advection–diffusion equation for this passive tracer is

\[
\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = \nabla \cdot (\kappa \nabla c), \tag{2}
\]

where \( c(x, y, t) \) is the concentration of the passive tracer and \( \kappa \) is the constant diffusivity. The numerical methods used to solve (2) are identical to those used to solve (1). However, the results to be presented here have quite different initial conditions on \( \xi \) and \( c \). The passive tracer \( c \) is always initialized as an axisymmetric and monotonic and is not necessarily axisymmetric.

3. Area coordinate transformation and effective diffusivity

Following Nakamura (1996) and Shuckburgh and Haynes (2003), here we present a brief derivation of the effective diffusion equation and resulting effective diffusivity diagnostic used in this study. To aid in the derivation, a diagram of the area coordinate is shown in Figure 2. Consider the transform from Cartesian \((x, y)\) coordinates to tracer \((C, s)\) coordinates, where \( C \) is a particular contour of the \( c(x, y, t) \) field and \( s \) is the position along that contour. Let \( dc \) be the differential element of \( C \) and \( ds \) be the differential element of \( s \). Let \( A(C, t) \) denote the area of the region in which the tracer concentration satisfies \( c(x, y, t) \geq C \), i.e.

\[
A(C, t) = \iint_{c \geq C} dx \, dy. \tag{3}
\]

Let \( \gamma(C, t) \) denote the boundary of this region. Note that \( A(C, t) \) is a monotonically decreasing function of \( C \) and that \( A(C_{\text{max}}, t) = 0 \). This guarantees a unique inverse function \( C(A, t) \) that is monotonic and decomposable into isolines. Now define \( u^C \) as the velocity of the contour \( C \), so that

\[
\frac{\partial c}{\partial t} + u^C \cdot \nabla c = 0. \tag{4}
\]

Noting that \( \nabla c / |\nabla c| \) is the unit vector normal to the contour, we can use (3) and (4) to write

\[
\frac{\partial A(C, t)}{\partial t} = \frac{\partial}{\partial t} \int_{c \geq C} dx \, dy = -\int_{\gamma(C, t)} u^C \cdot \nabla c \left| \frac{\nabla c}{|\nabla c|} \right| ds \tag{5}
\]

Using (2) in the last equality of (5), we obtain

\[
\frac{\partial A(C, t)}{\partial t} = \int_{\gamma(C, t)} \nabla \cdot (\kappa \nabla c) \left| \frac{\nabla c}{|\nabla c|} \right| ds - \int_{\gamma(C, t)} u \cdot \nabla c \left| \frac{\nabla c}{|\nabla c|} \right| ds. \tag{6}
\]

We now note that (since \( dx \, dy = ds \, dC' / |\nabla c| \))

\[
\frac{\partial}{\partial C} \iint_{c \geq C} (\ ) dx \, dy = \frac{\partial}{\partial C} \int_{c \geq C} ds \, dC' \left| \frac{\nabla c}{|\nabla c|} \right| = -\int_{\gamma(C, t)} \left( \right) ds \left| \frac{\nabla c}{|\nabla c|} \right|. \tag{7}
\]
where the minus sign on the right-hand side comes from our arbitrary definition of $A(C)$. Using (7) in (6) while noting that $\mathbf{u} \cdot \nabla = \nabla \cdot (\mathbf{c} \mathbf{u})$ because $\mathbf{u}$ is non-divergent, we obtain

$$\frac{\partial A(C, t)}{\partial t} = -\frac{\partial}{\partial C} \int_{c \leq C} \nabla \cdot (\kappa \nabla c) \, ds \frac{dC}{|\nabla c|} \left. \right|_{C}$$

$$+ \frac{\partial}{\partial C} \int_{c \leq C} \nabla \cdot (\mathbf{c} \mathbf{u}) \, ds \frac{dC}{|\nabla c|} \left. \right|_{C}$$

$$= -\frac{\partial}{\partial C} \int_{y(C,t)} \kappa |\nabla c| ds$$

$$+ \frac{\partial}{\partial C} \int_{y(C,t)} \mathbf{c} \cdot \nabla c \frac{dC}{|\nabla c|} ds.$$  

The third and fourth lines of (8) are obtained using the divergence theorem. The fourth line of (8) vanishes because the factor $c$ in the integrand can come outside the integral, leaving $\int_{y(C,t)} \mathbf{c} \cdot \nabla c \frac{dC}{|\nabla c|} ds$, which vanishes because $\mathbf{u}$ is non-divergent.

Since $A(C, t)$ is a monotonic function of $C$, there exists a unique inverse function $C(A, t)$. We now transform (8) from a predictive equation for $A(C, t)$ to a predictive equation for $C(A, t)$. This transformation is aided by

$$\frac{\partial A(C, t)}{\partial t} \bigg|_{C} = -\frac{\partial C(A, t)}{\partial t},$$

which, when used in (8), yields

$$\frac{\partial C(A, t)}{\partial t} = \frac{\partial C(A, t)}{\partial A} \frac{\partial}{\partial C} \int_{y(C,t)} \kappa |\nabla c| ds$$

$$+ \frac{\partial}{\partial C} \int_{y(C,t)} \kappa |\nabla c| ds.$$  

Because of (7), the integral $\int_{y(C,t)} \kappa |\nabla c| ds$ on the right-hand side of (10) can be replaced by $(\partial/\partial C) \int_{x \leq C} \kappa |\nabla c|^2 dx dy$. Then (10) can be written in the form

$$\frac{\partial C(A, t)}{\partial t} = \frac{\partial}{\partial A} \left( K_{\text{eff}}(A, t) \frac{\partial C(A, t)}{\partial A} \right),$$

where

$$K_{\text{eff}}(A, t) = \left( \frac{\partial C}{\partial A} \right)^{-2} \frac{\partial}{\partial A} \int_{c \leq C} \kappa |\nabla c|^2 dx dy.$$  

To summarize, the area coordinate has been used to transform the advection–diffusion equation (2) into the diffusion-only equation (11), in the process yielding the effective diffusivity $K_{\text{eff}}(A, t)$. Since $K_{\text{eff}}(A, t)$ can be computed from (12), it can serve as a useful diagnostic tool to help us understand the interplay of advection and diffusion in (2). However, note that, because of the use of $A$ as an independent variable, the effective diffusivity $K_{\text{eff}}(A, t)$ has the rather awkward units of $m^4 \text{s}^{-1}$. This is easily corrected by mapping the area coordinate into the equivalent radius coordinate, $r_e$, which is defined by $\pi r_e^2 = A$. Thus, transforming (11) to the equivalent radius using $2\pi r_e (\partial/\partial A) = (\partial/\partial r_e)$, we obtain

$$\frac{\partial C(r_e, t)}{\partial t} = \frac{\partial}{r_e} \left( r_e K_{\text{eff}}(r_e, t) \frac{\partial C(r_e, t)}{\partial r_e} \right),$$

where

$$K_{\text{eff}}(r_e, t) = \frac{K_{\text{eff}}(A, t)}{4\pi A}.$$  

Note that, with of the use of $r_e$ as an independent variable, the effective diffusivity $K_{\text{eff}}(r_e, t)$ has the units of $m^2 \text{s}^{-1}$. The effective diffusivity can also be interpreted in terms of the equivalent length of tracer contours (Nakamura, 1996), i.e.

$$L_e(r_e, t) = \left( \frac{K_{\text{eff}}(r_e, t)}{\kappa} \right)^{1/2} \frac{2\pi r_e}{k}.$$  

As shown by Nakamura (1996), $L_e$ reduces to the actual perimeter of the tracer contour when $|\nabla c|$ is constant along that contour, thereby reducing (15) to $k = K_{\text{eff}}$. During differential advection, $L_e$ can become quite large as tracer contours are stretched and folded, exposing more interface for diffusion to act.

The effective diffusivity diagnostics $K_{\text{eff}}(A, t)$, $K_{\text{eff}}(r_e, t)$ and $L_e(r_e, t)$ can be calculated at a given time $t$ from the output $c(x, y, t)$ of the numerical solution of (2). The calculation of $K_{\text{eff}}(A, t)$ involves the following discrete approximation of the right-hand side of (12). First, the desired number of area coordinate points is chosen ($n_A = 200$ for the results shown here). The tracer contour interval is set using $\Delta C = |\max(c) - \min(c)|/n_A$. Next, $|\nabla c|^2$ is calculated at each model grid point. Then, a discrete approximation of the function $A(C, t)$ is determined by adding up the area within each chosen C contour, i.e. by using a discrete approximation to (3). The discrete approximation to $A(C, t)$ is then converted to a discrete approximation of its inverse, $C(A, t)$. The denominator of the effective diffusivity diagnostic, $(\partial C/\partial A)^2$, is calculated by taking second-order finite differences of $C(A, t)$. The numerator of the right-hand side of (12) is then calculated in the same manner, which completes the calculation of the effective diffusivity $K_{\text{eff}}(A, t)$. The remaining effective diffusivity diagnostics $K_{\text{eff}}(r_e, t)$ and $L_e(r_e, t)$ are then easily computed using (14)–(15). As will be shown, plots of these diagnostics reveal the locations of partial barrier and mixing regions within the vortex.

In the next section, two-dimensional plots of effective diffusivity will be shown. This can be done because effective diffusivity is constant along a tracer contour, and tracer contours meander in $(x, y)$ space.

\footnote{It is important to note that $r_e$ is a coordinate based on vorticity rearrangement, and therefore it does not necessarily correspond exactly to the actual radius from the centre of the flow structure. However, it will be shown to be a reasonable approximation for our idealized cases here, because the flow is mostly circumferential and has a natural centre.}
From another point of view, $\kappa_{\text{eff}}(r_c, t)$ can be mapped to $\kappa_{\text{eff}}(x, y, t)$ because each horizontal grid point is associated with an equivalent radius. Note, however, that by using the area coordinate the dimensionality has been reduced, so that these plots are effectively only showing one-dimensional information.

4. Pseudospectral model experiments and results

We now use the effective diffusivity diagnostic to understand the transport and mixing properties of a number of flows believed relevant to hurricane dynamics. The cases selected here are (i) an elliptical vortex, (ii) a Rankine-like vortex embedded in a random turbulent vorticity field and (iii) unstable vorticity rings. In the first two cases, the initial vorticity is maximum at the vortex centre, which is more characteristic of tropical storms rather than fully developed hurricanes. In the third case, the initial vorticity is maximum in the ring, which is characteristic of strong or intensifying hurricanes. All of the experiments are unforced and exhibit properties of two-dimensional turbulence, in particular the preferential decay of enstrophy over kinetic energy. In the following subsections, the initial condition and parameters for each experiment are shown, and the results are presented and discussed.

4.1. Elliptical vortex

The initial elliptical vortex is constructed in a manner similar to Guinn (1992). In polar coordinates, the initial vorticity field is specified by

\[
\zeta(r, \phi) = \zeta_0 \begin{cases} 
1, & 0 \leq r \leq r_i \alpha(\phi), \\
1 - f_\alpha(r'), & r_i \alpha(\phi) \leq r \leq r_o \alpha(\phi), \\
0, & r_o \alpha(\phi) \leq r,
\end{cases}
\]

(16)

where $\alpha(\phi) = [(1 - \epsilon^2)/(1 - \epsilon^2 \cos^2(\phi))]^{1/2}$ is an ellipticity augmentation factor for the ellipse $(x/a)^2 + (y/b)^2 = 1$ (where $a$ is the semi-major axis and $b$ is the semi-minor axis) with eccentricity $\epsilon = (1 - (b^2/a^2))^{1/2}$. Here, $\zeta_0$ is the maximum vorticity at the centre, $f_\alpha(r') = \exp(-\lambda/r') \exp[1/(r' - 1)]$ is a monotonic shape function with transition steepness parameter $\lambda$, $r' = [r - r_o \alpha(\phi)]/[r_o \alpha(\phi) - r_i \alpha(\phi)]$ is a non-dimensional radius, and $r_i$ and $r_o$ are the radii where the vorticity begins to decrease and where it vanishes, respectively. For the special case of $\alpha(\phi) = 1$ the field is axisymmetric. For the experiment conducted, $\zeta_0 = 5.0 \times 10^{-3}$, $\lambda = 2.0$, $\epsilon = 0.70$, and $r_i$ and $r_o$ were set to 30 and 60 km, respectively. The initial tracer field was Gaussian: $\hat{c}(r) = c_0 \exp(-r/r_o)^2$, with $c_0 = 1000$ and $r_o = 50$ km.

Plots of vorticity and effective diffusivity $\kappa_{\text{eff}}$ at $t = 1.33$ h during the evolution of the elliptical vortex are shown in Figure 3. At this time, two filaments of high vorticity associated with breaking VRWs are clearly visible. Associated with these filaments are regions of large effective diffusivity. The effective diffusivity peaks just upwind of the filaments and extends further upwind. The main vortex acts as a transport barrier during the filamentation. In terms of an arbitrary passive tracer, these results indicate that the tracer will tend to be well-mixed horizontally in the wave-breaking surf zone, while tracers initially in the vortex core will be trapped there. During its evolution, continued wave-breaking episodes occur as the ellipse tries to axisymmetrize. However, axisymmetrization is not complete for $t \leq 48$ h, and the surf zone is a robust feature throughout the entire simulation (not shown). The ability of an elliptical vortex to axisymmetrize (Melander et al., 1987) via inviscid dynamics was shown to be determined by the sharpness of its edge (Koumoutsakos, 1997; Dritschel, 1998). If the vortex is more Rankine-like (i.e. possessing a sharp edge), it will tend to rotate and not generate filaments. If, on the other hand, the transition is not sharp, there will be a tendency to generate filaments and axisymmetrize.

![Figure 3](image-url)
Although it occurs on much smaller time- and length-scales, there is an analogy between this surf zone in tropical cyclones and the planetary Rossby-wave-breaking surf zone associated with the wintertime stratospheric polar vortices (McIntyre and Palmer, 1983, 1984, 1985; Juckes and McIntyre, 1987; McIntyre, 1989; Bowman, 1993; Waugh et al., 1994). Planetary waves excited in the troposphere may propagate vertically and cause wave breaking to occur on the edge of the stratospheric polar vortex, from which chemical constituents can be mixed into the midlatitudes. The wintertime stratospheric polar vortices display similar processes to our experiment: namely, the core vortex is a transport barrier and the surf zone is a chaotic mixing region. The existence of the main vortex barrier was thought to be due to the strong PV gradient, a restoring mechanism for perturbations imposed upon it. Rossby-wave breaking has also been examined in more idealized frameworks (see, for example, Polvani and Plumb, 1992; Koh and Plumb, 2000).

In tropical cyclones, the deformation of an initially circular vortex core to an ellipse may happen due to external (e.g. vertical shear) or internal (e.g. PV generation by asymmetric moist convection) processes. The relaxation to axisymmetry will produce wave-breaking episodes and, as we have shown here, moderate mixing regions in the associated surf zone.

4.2. Rankine vortex in a turbulent vorticity field

A Rankine vortex in a stirred vorticity field may be represented mathematically by

\[
\begin{align*}
\zeta(x, y, 0) &= \zeta_m \left\{ \begin{array}{ll}
1, & 0 \leq r \leq r_1, \\
S \left( \frac{r-r_2}{r_2-r_1} \right), & r_1 \leq r \leq r_2, \\
0, & r_2 \leq r,
\end{array} \right. \\
&+ \zeta_{\text{rand}}(x, y) \left\{ \begin{array}{ll}
1, & 0 \leq r \leq r_3, \\
S \left( \frac{r-r_4}{r_4-r_3} \right), & r_3 \leq r \leq r_4, \\
0, & r_4 \leq r,
\end{array} \right.
\end{align*}
\]

(17)

where \( \zeta_m = 5 \times 10^{-3} \) s\(^{-1} \) is the maximum vorticity of the Rankine vortex, \( S(x) = 1 - 3x^2 + 2x^3 \) is a cubic polynomial shape function providing smooth transitions from \( r_1 = 20 \) km to \( r_2 = 30 \) km and from \( r_1 = 120 \) km to \( r_3 = 180 \) km, and \( \zeta_{\text{rand}}(x, y) \) is a spatially random turbulent vorticity field given by

\[
\zeta_{\text{rand}}(x, y) = \sum_{k=-k_{max}}^{k_{max}} \sum_{\ell=-\ell_{max}}^{\ell_{max}} \zeta_{k, \ell} e^{i(2\pi/L)(kx+\ell y)}.
\]

Here, \( k_{max} \) and \( \ell_{max} \) are the spectral truncation limits in \( x \) and \( y \), \( L \) is the domain length, \( \zeta_{k, \ell} \) is random with a maximum amplitude of \( 1.5 \times 10^{-3} \) s\(^{-1} \) and the total wavenumber \( k = (k^2 + \ell^2)^{1/2} \) is set for spatial scales primarily between 20 and 40 km. The initial tracer field was linear: \( \bar{c}(r) = c_0 + ar \), where \( c_0 = 1000 \) and \( a = -4.422 \) km\(^{-1} \).

As an analogy to real tropical cyclones, the Rankine-like vortex can be thought of as the tropical cyclone core and the stirred vorticity field can be thought of as generated by random convection. The initial condition for this experiment is shown in the top panel of Figure 4. As the simulation evolves, the core vortex begins to axisymmetrize the random vorticity elements. At \( t = 9.5 \) h the core vortex starts to become a partial barrier region. Also note that a thin partial barrier ring begins to form at the radius of maximum winds just outside the central vorticity core. Outside the vortex core, chaotic mixing is occurring as the random vorticity anomalies are being axisymmetrized by the shearing flow. By \( t = 40 \) h (bottom panels) the relative vorticity exhibits a central monopole, a low-vorticity moat and a secondary ring of enhanced vorticity. Comparing the two bottom panels, the low-vorticity moat coincides with the ring of moderate effective diffusivity (100 \( \leq \kappa_{\text{eff}} \leq 500 \) m\(^2\) s\(^{-1} \)). In real tropical cyclones, the moat region is a region of suppressed convective activity due to the combined effects of subsidence (induced by return flow from the eyewall) and strain-dominated flow (Rozoff et al., 2006). The moat here was identified as a region of enhanced mixing. The secondary ring of enhanced vorticity coincides with the ring of low effective diffusivity (\( \kappa_{\text{eff}} \leq 100 \) m\(^2\) s\(^{-1} \)). The azimuthal mean wind (not shown) associated with the bottom left panel of Figure 4 has two maxima. The first is the primary azimuthal jet located at the edge of the central vorticity monopole, and the second is the secondary azimuthal jet that occurs at the outer edge of the secondary ring of enhanced vorticity. In the effective diffusivity plot, these jets are partial barriers (white rings) with \( \kappa_{\text{eff}} \leq 100 \) m\(^2\) s\(^{-1} \). Therefore, in this idealized experiment, azimuthal jets in hurricanes are shown to be partial transport barriers, resistant to radial mixing.

4.3. Unstable vorticity rings

Five experiments were conducted for different dynamically unstable hurricane-like vortices. The initial vorticity field consists of a vorticity ring (the eyewall) and a relatively low-vorticity centre (the eye). Observations (Kossin and Eastin, 2001; Mallen et al., 2005) indicate that strong or intensifying hurricanes are often characterized by such vorticity fields. The average vorticity over the inner core was set to be \( \omega_0 = 2.0 \times 10^{-3} \) s\(^{-1} \), corresponding to a peak tangential wind of approximately 40 m\ s\(^{-1} \) in each case.

The initial condition on the vorticity is given in polar coordinates by \( \zeta(r, \phi) = \zeta(r) + \zeta'(r, \phi) \), where \( \zeta(r) \) is an axisymmetric vorticity ring defined by

\[
\zeta(r) = \frac{\zeta_1}{S \left( \frac{r-r_1}{r_1} \right)} + \zeta_2 S \left( \frac{r-r_2}{r_2-r_1} \right), \quad 0 \leq r \leq r_1,
\]

\[
\zeta_2 S \left( \frac{r-r_1}{r_2-r_1} \right) + \zeta_3 S \left( \frac{r-r_2}{r_3-r_2} \right), \quad r_1 \leq r \leq r_2,
\]

\[
\zeta_3 S \left( \frac{r-r_2}{r_3-r_2} \right) + \zeta_4 S \left( \frac{r-r_3}{r_4-r_3} \right), \quad r_2 \leq r \leq r_3,
\]

\[
\zeta_4 S \left( \frac{r-r_3}{r_4-r_3} \right), \quad r_3 \leq r \leq r_4,
\]

\[
\zeta_{\text{rand}}(x, y), \quad r_4 \leq r \leq \infty.
\]

(19)
Figure 4. The initial vorticity field (a) and side-by-side panels ((b)–(e); at 9.5 h and 40 h) of relative vorticity and effective diffusivity for the Rankine-like vortex in a turbulent vorticity field. The model domain is 600 km by 600 km, but only the inner 200 km by 200 km is shown.
where \(\zeta_1, \zeta_2, \zeta_3, r_1, r_2, r_3\) and \(r_4\) are constants, and \(S(x)\) is the cubic polynomial interpolation function defined previously. The eyewall is defined as the region between \(r_2\) and \(r_3\). Schubert et al. (1999) defined two parameters to describe these hurricane-like vorticity rings: a ring thickness parameter \(\delta = (r_1 + r_2)/(r_3 + r_4)\) and a ring hollowness parameter \(\gamma = \zeta_1/c_{av}\). The parameters used for each of the five experiments are listed in Table I. Note that \(\zeta_2\) is set to be slightly negative so that the domain-averaged vorticity is equal to zero. Each ring is perturbed with a broadband impulse of the form

\[
\zeta'(r, \phi, 0) = \zeta_{amp} \sum_{m=1}^{8} \cos(m\phi + \phi_m) \begin{cases} 
0, & 0 \leq r \leq r_1, \\
S \left(\frac{r_2-r_1}{r_3-r_1}\right), & r_1 \leq r \leq r_2, \\
1, & r_2 \leq r \leq r_3, \\
S \left(\frac{r-r_3}{r-r_3}\right), & r_3 \leq r \leq r_4, \\
0, & r_4 \leq r \leq \infty,
\end{cases}
\]

where \(\zeta_{amp} = 1.0 \times 10^{-5} \text{s}^{-1}\) is the amplitude and \(\phi_m\) the phase of azimuthal wavenumber \(m\). For this set of experiments, the phase angles \(\phi_m\) were chosen to be random numbers in the range \(0 \leq \phi_m \leq 2\pi\). In real hurricanes, such asymmetries are expected to develop from a wide spectrum of background turbulent and convective motions. The initial tracer field used was an axisymmetric cone: \(\tilde{c}(r) = c_0 + ar\), where \(c_0 = 1000\) and \(a = -4.422 \text{km}^{-1}\).

Two simulations from Table I are illustrated. The first (Exp. A) is a thick, filled ring, while the second (Exp. D) is a thin, hollow ring. According to Schubert et al. (1999), as the rings become thicker and filled disturbance growth rates become smaller and at lower wavenumber. As the rings become very thin and hollow, they rapidly break down and sometimes evolve into persistent mesovortices (Kossin and Schubert, 2001). Experiment A is shown in Figure 5. At \(t = 13.0\text{h}\) (middle left panel), the ring is breaking down at azimuthal wavenumber \(m = 4\), giving the appearance of a polygonal eyewall with straight-line segments. Note that both the inner and outer VRWs are breaking at this time. The breaking of the inner VRW has allowed vorticity to be pooled into four regions. In the effective diffusivity plot (middle right panel) there are two distinct radial regions of mixing, separated by a rather strong, thin barrier region. The inner mixing region is at the location of the inner breaking VRW, while the outer mixing region exists at the location of the outer breaking VRW. The waves are phase-locked and helping each other grow, resulting in radial air movement and mixing. During this time the passive tracer field becomes relatively well mixed in the radial intervals of the VRW activity. However, the initial gradient is maintained in the barrier region in between (not shown). Progressing to \(t = 41.0\text{h}\), the magnitude of the mixing due to the wave activity is smaller, but the barrier region remains intact.

The breakdown of the ring in Exp. D is shown in Figure 6. The disturbance growth rates are much larger in this case, allowing the ring to break down much faster. Multiple mesovortices initially form (middle left panel). During the formation stage, these mesovortices and associated filamentary structures are characterized as strong mixing regions by the diagnostic (middle right panel of Figure 6). This was a surprising result, since coherent vortices are known to be partial barriers. However it is possible that the enhanced effective diffusivity is a result of mixing that is occurring at smaller scales inside the mesovortices just after their formation (however there is some uncertainty whether this is real or an artefact of the choice of the initial tracer profile—see the discussion in section 5.2). The mesovortices persist for a very long time, and at \(t = 20.0\text{h}\) there are three mesovortices left after some mergers have occurred. At this time, the passive tracer has been homogenized locally inside the mesovortices, causing low effective diffusivities there. It is also important to note that the appearance that the vorticity field is more compact than the effective diffusivity field in these plots is due to the arbitrary cut-off contour of the vorticity, \(5.0 \times 10^{-4} \text{s}^{-1}\).

There is vorticity mixing occurring outside the mesovortices contributing to large effective diffusivities in, for example, the bottom right panel of Figure 5. Based on these results, in conjunction with the Rankine-like vortex in a turbulent vorticity field, we surmise that barotropic geophysical vortices of all horizontal scales tend to act as partial barrier regions when they are long-lived; however, strong mixing can occur in their former stages.

To illustrate further the two regimes of internal mixing, Hovmöller plots of \(\kappa_{eff}(r, t)\) are shown in Figure 7 for Experiments A and D. For the A ring (left panel), there exist two distinct mixing regions at \(20\text{km} \leq r_e \leq 30\text{km}\) and \(40\text{km} \leq r_e \leq 55\text{km}\). These mixing regions are associated with the counter-propagating VRWs evident in the middle panels of Figure 5. For the D ring, in which a rapid breakdown occurs, the entire hurricane inner core (\(10\text{km} \leq r_e \leq 60\text{km}\)) is a chaotic mixing region. These two types of mixing regimes are further clarified in Figure 8, which shows the time-averaged effective diffusivity \(\kappa_{eff}\) for all five rings. For the rings with slower growth rates (A and B) there exist two peaks in \(\kappa_{eff}(r_e)\) coincident with inner and outer VRW activity. For the rings with faster growth rates (C, D and E), the entire inner core is a chaotic mixing region.
Figure 5. The initial vorticity field (a) and side-by-side panels (b)–(e); at 13 h and 41 h of relative vorticity and effective diffusivity $\kappa_{\text{eff}}$ for a prototypical thick, filled unstable vorticity ring (Experiment A of Table I).
Figure 6. The initial vorticity field (a) and side-by-side panels (b)–(e); at 6 h and 20 h) of relative vorticity and effective diffusivity $\kappa_{\text{eff}}$ for a prototypical thin, hollow unstable vorticity ring (Experiment D of Table I).
During the evolution of each ring, the radius of maximum wind varies, but it is generally confined to radii between 30 and 40 km. Thus, for thick, filled rings the hurricane tangential jet acts as a partial barrier region for $t \leq 48$ h, while for thin, hollow rings the hurricane tangential jet breaks down and turbulent mixing in the entire inner core ensues. The implication of this result for real hurricanes is that if the eyewall is very thick, passive tracers will not easily be mixed across the eyewall during barotropic instability, but may be mixed between the eye–eyewall and environment–eyewall by the inner and outer breaking VRWs, respectively. If, on the other hand, the eyewall is thin, as in rapidly intensifying hurricanes (Kossin and Eastin, 2001), passive tracers can be mixed across the eye, eyewall, and environment, and at a much faster rate. Assuming hurricanes have a maximum of equivalent potential temperature ($\theta_e$) at low levels in the eye, our results indicate that the inner, breaking, VRW will mix air parcels with high $\theta_e$ into the eyewall, supporting intensification and the hurricane superintensity mechanism concept (Persing and Montgomery, 2003). This mixing will be more rapid for the breakdown of thin rings.

The mixing regime in which the tangential jet acts as a partial barrier is analogous to the results of Bowman and Chen (1994), who found that air poleward of a barotropically unstable stratospheric jet remained nearly perfectly separated from midlatitude air. Our hurricane results are again analogous to planetary-scale mixing, and it appears that under certain conditions azimuthal jets in hurricanes can become asymmetric but still remain partial (but leaky) barriers to radial mixing.

One may wonder how these results generalize to the entire $\delta$–$\gamma$ plane, which covers all possible annular vorticity structures in two dimensions. From a vorticity dynamics perspective, Hendricks et al. (2009) showed that the most vigorous mixing episodes occurred with thin, hollow rings, while only minor mixing occurred in thick and filled rings. These results would probably generalize in a similar manner, with complete mixing between the eye, eyewall, and local environment occurring for thin, hollow rings, and partial barriers existing for thicker and more filled rings. Interestingly, that study documented large pressure drops for the barotropic instability of thin, hollow rings. Considering that $\kappa_{\text{eff}}$ is approximately twice as large for breakdowns of thin rings as for thick rings (4000 versus 2000 m$^2$ s$^{-1}$, respectively, Figure 8), the high $\theta_e$ air in the low-level eye may be mixed into the eyewall at a much faster rate for thin rings, thereby accentuating the intensification process.

5. Sensitivity tests

In order to assess the robustness of effective diffusivity as a diagnostic of the mixing properties of a flow, a number of sensitivity tests were conducted: (i) tracer diffusion coefficient, (ii) initial tracer distribution and (iii) the accuracy of the discrete approximation to the diagnostic (12).

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Figure 7. Hovmöller plots depicting the temporal evolution of $\kappa_{\text{eff}}(r_e, t)$ for the A ring (left) and the D ring (right).

Figure 8. Time-averaged (0–48 h) effective diffusivity as a function of equivalent radius for all unstable rings. The radius of maximum wind varies during the evolution but generally lies in the region between 30 and 40 km.
5.1. Tracer diffusion coefficient

In the area-based coordinate system, it is expected that the effective diffusivity will increase with increasing tracer diffusivity $\kappa$. As material lines are stretched and folded there exists more interface for diffusion to produce irreversible mixing, and if the diffusion coefficient is larger then the level of mixing should be larger, as area can diffuse faster between tracer contours. This is clearly illustrated in Figure 9 for the unstable ring experiment C. This experiment is similar to Experiment D in that a large radial segment becomes a chaotic mixing region (Figure 7, right panel). Four different values of the tracer diffusivity are chosen: $\kappa = 50, 25, 10$ and 0.1 m$^2$ s$^{-1}$. The larger tracer diffusivities clearly have larger effective diffusivities, and the radial character of the profiles is broadly preserved for each case. For example, the $\kappa = 50, 25$ and 10 m$^2$ s$^{-1}$ cases are able to capture the peak effective diffusivity at $r_e = 30$ km. The $\kappa = 0.1$ m$^2$ s$^{-1}$ case is not seen on the figure because the peak effective diffusivity associated with it is only $\kappa_{eff} = 20$ m$^2$ s$^{-1}$, too low to be visible with the plot scaling. The same plot is shown in Figure 10 for the normalized effective diffusivity $\kappa_{eff}(r_e, t)/\kappa$. Note that the normalized effective diffusivity is not very sensitive to varying $\kappa$, and therefore it is the best measure of chaotic advection.

5.2. Initial tracer distribution

Since effective diffusivity maps out the mixing properties of a flow, it is supposed to be mostly insensitive to the initial tracer field, provided it is monotonic and well behaved. In order to illustrate this, plots of effective diffusivity versus equivalent radius are shown in Figure 11 for three different initial axisymmetric tracer fields: a Gaussian distribution with maximum value of 1000 (used in the elliptical vorticity field and Rankine-like vortex in a turbulent vorticity field) and linearly decreasing distributions with maximum values of 1000 (used in the unstable vorticity ring experiments) and 5000. Each of these curves has different $dC/dr_e$ (or $dC/DA$), used in the denominator of the effective diffusivity diagnostic. The $\kappa_{eff}$ profiles are almost identical for the two linear cases, and only a slight variation is found for the Gaussian case. The Gaussian case departs from the linear cases slightly at small radii. The likely reason for this is that the slope of the tangent line ($dC/dr_e$) is very small there, causing the effective diffusivity diagnostic to be unrealistically distorted. We feel that the linearly decreasing initial tracer profile is the best to use because it guarantees constancy of the initial $dC/DA$ in the domain. Overall, effective diffusivity is found to be mostly insensitive to the initial tracer profile, provided it is monotonic.

The sensitivity of these results to non-monotonic tracer profiles was also examined. Three additional experiments were conducted for unstable rings A and D and the Rankine vortex in which the initial tracer field was exactly the same as the initial vorticity field. For experiment A, the partial barrier in the middle right panel of Figure 5 still existed, however there also existed a thin polygonal region where $\kappa_{eff}$ was large on the inner side of this barrier region. The four inner vorticity pools had elevated effective diffusivity, consistent with the control experiment. For experiment D, the mesovortices at $t = 6$ h (middle panels of Figure 6) had anomalously...
low (rather than high) effective diffusivities, consistent with previous findings of small Lyapunov exponents in coherent vortices. Considering this finding, there exists uncertainty in whether these mesovortices are partial barriers or mixing regions at this time. At the later time \( t = 20 \) h in this experiment, the mesovortices were identified as partial barriers, consistent with the control experiment (bottom right panel of Figure 6). For the Rankine vortex in a turbulent vorticity field, the coarse-grained results were largely unchanged, in that the vortex core became a partial barrier in time while mixing was occurring outside. However, at \( t = 40 \) h this experiment produced a stronger mixing region at the location of the moat and it also did not capture the partial barrier region at the location of the secondary wind maximum. Based on these results, there exists a rather strong sensitivity of effective diffusivity to the initial tracer field if it is not monotonic and broadly distributed. However, in these cases this method also caused \( dC/dA \) in (12) to be very small outside the vorticity cores. For the unstable rings, this created physically unrealistic large effective diffusivities there, which is clearly wrong since those tracer contours were not disturbed. These results are consistent with Shuckburgh and Haynes (2003), who found a strong sensitivity to the initial tracer profile in regions where the spatial gradient was approximately zero (see their section III.B.2). The conclusion of these tests is that, while it may be desirable to make the initial tracer field exactly the same as the initial vorticity field, ultimately in certain instances (such as the vorticity configurations here) the effective diffusivity diagnostic is not as reliable macroscopically for the reasons stated above.

5.3. Number of area points

Sensitivity tests were performed using varying numbers of area points in the discrete approximation to the effective diffusivity diagnostic (12). To illustrate the sensitivity to the discrete approximation, the effective diffusivity versus equivalent radius is shown in Figure 12 for the C unstable ring with \( n_A = 50 \) and 200 (used in all the experiments in this article). Both curves produce peaks near \( r_e = 30 \) km, but only the \( n_A = 200 \) curve captures the peak value of \( k_{eff} = 5000 \text{m}^2\text{s}^{-1} \) at \( r_e \approx 35 \) km and the dip at \( r_e = 40 \) km. The conclusion is that our choice \( n_A = 200 \) is sufficient to resolve the mixing variability of the inner core for our chosen model resolution.

6. Conclusions

The two-dimensional mixing properties of non-divergent barotropic flows resembling the idealized evolution of both tropical storms and hurricanes were quantified using the effective diffusivity diagnostic. The location and magnitude of both turbulent mixing and partial barrier regions were identified, yielding insight into how passive tracers are asymmetrically mixed at low levels. The primary finding is that breaking vortex Rossby waves (VRWs), resulting from either axisymmetrization or dynamic instability, are quite effective at mixing passive tracers over large horizontal distances in hurricanes, and that this mixing is likely an important internal mechanism of intensity change.

For monotonic vortices that have profiles similar to tropical storms, the wave breaking and mixing was generally confined to outside of the radius of maximum wind. For the elliptical vortex, a 20–30 km wide surf zone existed which was characterized by turbulent mixing. For the Rankine vortex inside a convective vorticity field, strong mixing occurred as the random vorticity elements were axisymmetrized. In both of these cases, the centre of the storm was a partial barrier, or containment vessel, and air was not easily mixed with the local environment. For unstable rings, which are analogous to strong or intensifying hurricanes, both the inner and outer counter-propagating VRWs break due to reinforcement from barotropic instability, causing two mixing regions: one between the eye and eyewall and one between the eyewall and local environment. In the case of thick rings, the disturbance growth rates are small and a long-lived asymmetric partial barrier region may exist between the two breaking waves, coincident with the tangential jet. In the case of thin rings that are very dynamically unstable, the rapid breakdown created a strong chaotic mixing region over the entire hurricane inner core (eye, eyewall and local environment). In this case, passive tracers may be horizontally mixed over large radial distances (approximately 60–80 km in our experiments). Since observations show a maximum of \( \theta_c \) at low levels in the eye (Eastin et al. 2002a, 2002b), our results indicate that the inner, breaking VRW would be quite effective at mixing air parcels with high \( \theta_c \) into the eyewall, causing intensification. For thick rings, the centre of the hurricane eye remains a partial barrier during barotropic instability because turbulent mixing associated with the inner, breaking VRW is confined to the outer eye. In these cases, it is possible that the highest \( \theta_c \) air may never be mixed into the eyewall, limiting the level of intensification via internal mixing.

Both primary and secondary azimuthal wind maxima were identified as partial barriers in our simulations.
These jets act as mixing barriers because they are located near regions of strong radial PV gradients (cf. McIntyre, 1989). A surprising result is that the primary jet barrier region can be maintained for long times during barotropic instability. Additionally, it can maintain itself as a partial barrier when it is deformed asymmetrically to a polygon with straight-line segments. Therefore, in this simple framework the hurricane primary azimuthal jet appears to be a robust transport barrier for both dynamically stable and unstable vortices, provided the latter vortices are characterized by thick annular vorticity structures. Whether or not this jet is such a robust barrier in real hurricanes, where moist processes and environmental vertical shear are active, is an open question.

One way to extend the present work would be to use effective diffusivity as a diagnostic for transport and mixing in axisymmetric hurricane models (Rotunno and Emanuel, 1987; Hausman et al., 2006). Of particular interest would be examining the vertical structure of mixing between the eye and the eyewall. The diagnostic could also be used to understand aspects of the transport and mixing of water vapour in the frictional boundary layer of hurricanes.

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