Chebyshev Spectral Methods for Limited-Area Models. Part II: Shallow Water Model

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ABSTRACT

Chebyshev spectral methods were studied in Part I for the linear advection equation in one dimension. Here we extend these methods to the nonlinear shallow water equations in two dimensions. Numerical models are constructed for a limited domain on a β-plane, using open (characteristic) boundary conditions based on Riemann invariants to simulate an unbounded domain. Reflecting boundary conditions (wall and balance) are also considered for comparison. We discuss the formulation of the Chebyshev-tau and Chebyshev-collocation discretizations for this problem. The tau discretization avoids aliasing error in evaluating quadratic nonlinear terms, while the collocation method is simpler to program.

Numerical results from a linearized one-dimensional test problem demonstrate that with the characteristic boundary conditions the stability properties for various explicit time differencing schemes are essentially the same as obtained in Part I for the linear advection equation. These open boundary conditions also give much more accurate results than the reflecting boundary conditions. In two dimensions, numerical results from the nonlinear models indicate that the Chebyshev-tau discretization should be based on the rotational form of the equations for efficiency, while the Chebyshev-collocation discretization should be based on the advective form for accuracy. Little difference is seen between the tau and collocation solutions for the test cases considered, other than efficiency: with explicit time differencing, the collocation model requires an order of magnitude less computer time.

1. Introduction

A limited-area model is an attempt to simulate or forecast the conditions in a limited region of the atmosphere, while in some sense recognizing that the real atmosphere has no lateral boundaries. Although such models usually are discretized in space using finite differences, the spectral approach offers the hope of obtaining much higher accuracy and possibly much higher efficiency. In Part I of this study (Fulton and Schubert, 1987; referred to hereafter simply as Part I) we demonstrated that spectral methods based on Chebyshev polynomial expansions may be a good choice for such limited-area models. However, in that study we considered only the simple one-dimensional linear advection equation, whereas most physical situations worth modeling are much more complicated, usually involving coupled systems of nonlinear equations with nontrivial boundary conditions.

The purpose of this paper is to examine the formulation and performance of Chebyshev spectral methods for a prototype model more representative of actual meteorological models. Our particular interest is in limited-area models based on primitive equations; to avoid the added complexity of vertical structure while retaining many of the essential features of that

system, we study here the nonlinear shallow water equations on a limited domain. We include no parameterized physics: rather, we use specified forcing and concentrate on the ability of the Chebyshev methods to represent the dynamics properly. In section 2 we discuss the governing equations and present some suitable boundary conditions. Chebyshev-tau and Chebyshev-collocation methods for this model are formulated in section 3. In section 4 we present numerical results from these Chebyshev models. Our conclusions are summarized in section 5.

2. Governing Equations and Boundary Conditions

In Cartesian (x, y) coordinates the nonlinear shallow water equations are

$$
\left.
\begin{array}{c}
\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} - fv + \dfrac{\partial \phi}{\partial x} = F \\[2ex]
\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + fu + \dfrac{\partial \phi}{\partial y} = G \\[2ex]
\dfrac{\partial \phi}{\partial t} + u\dfrac{\partial \phi}{\partial x} + v\dfrac{\partial \phi}{\partial y} + (\bar{\phi} + \phi)\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) = Q
\end{array}
\right\} . \quad (2.1)
$$

Here u and v are the velocity components in the x and y directions, respectively, ϕ is the deviation of the geopotential (free surface height h times the acceleration g due to gravity) from the constant positive reference

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value $\bar{\phi}$, f is the Coriolis parameter, and F, G and Q represent specified forcing. This *advective* form of the model is equivalent to the *rotational* form

$$\left.\begin{array}{l}\dfrac{\partial u}{\partial t}-\left(f+\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\right)v+\dfrac{\partial}{\partial x}\left[\phi+\dfrac{1}{2}(u^2+v^2)\right]=F\\[3mm]\dfrac{\partial v}{\partial t}+\left(f+\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\right)u+\dfrac{\partial}{\partial y}\left[\phi+\dfrac{1}{2}(u^2+v^2)\right]=G\\[3mm]\dfrac{\partial\phi}{\partial t}+\dfrac{\partial}{\partial x}[(\bar{\phi}+\phi)u]+\dfrac{\partial}{\partial y}[(\bar{\phi}+\phi)v]=Q\end{array}\right\}.$$

(2.2)

Both (2.1) and (2.2) can be used as starting points for developing numerical models. We will consider both f-plane (constant f) and β-plane ($f = f_0 + \beta y$ with f_0 and β constant) models.

The solutions of the shallow water model on an infinite domain can be characterized in terms of the corresponding normal modes, i.e., the eigenfunctions of the spatial operator of the model. Since these modes form a complete set, the solution of the model for particular data (i.e., initial conditions and forcing) can be expanded in terms of them with expansion coefficients which depend on time. In the unforced linear case, the expansion coefficients oscillate sinusoidally in time at the frequency given by the corresponding eigenvalue, and hence the solution is a superposition of waves. This correspondence with waves allows us to classify each normal mode as either a gravity mode (more properly, an inertia–gravity mode) or a Rossby mode (a geostrophic mode if f is constant). Gravity and Rossby modes are referred to as "fast" and "slow" modes, respectively, based on the relative sizes of the corresponding eigenvalues.

In the forced and/or nonlinear case, the time dependence of the expansion coefficients is more general. The part of the solution composed of the slow (Rossby) modes and those components of fast (gravity) modes which are evolving slowly with time will be referred to here as the balanced flow, since it is close to the solution of a corresponding balanced model (e.g., a quasi-geostrophic model). In many situations of meteorological interest, this balanced flow contains most of the energy; the remainder of the solution, consisting of rapidly progapating gravity waves, contains relatively little energy. Nevertheless, simplifying the equations to form a balanced model by eliminating terms responsible for the propagating gravity waves is often not desirable, since this also distorts the balanced flow.

To solve the shallow water model on a limited region of the (x, y)-plane one must specify boundary conditions. These should be chosen so that the resulting problem is mathematically well posed: the problem should have a unique solution for any combination of data (initial conditions and forcing), and this solution should depend continuously on the data. Furthermore, the boundary conditions should be physically reason-

able. Here we are particularly interested in simulating a portion of an infinite domain with a limited-area model and hence seek boundary conditions which minimize the impact of the computationally imposed boundaries. In light of the discussion above, ideal boundary conditions would allow the balanced flow in the model to closely approximate that of the solution on an infinite domain, and allow the outward-propagating gravity waves to leave the model domain without reflecting.

Open boundary conditions, i.e., those which transmit at least some portion of the waves incident on the boundary, have been studied extensively. Such conditions are generally based on the Sommerfeld radiation condition (e.g., Pearson, 1974; Orlanski, 1976; Hack and Schubert, 1981) or the related Riemann invariants (Wurtele et al., 1971; Elvius and Sundström, 1973), both of which are discussed in Courant and Hilbert (1962, pp. 315, 430). For dispersive systems or problems in more than one space dimension such conditions are inexact. Higher-order approximations have been obtained (Engquist and Majda, 1977; Bayliss and Turkel, 1980) at the cost of increased complexity. Exact conditions have also been obtained (Bennett, 1976) but are impractical to implement. Israeli and Orszag (1981) have obtained good results by combining open boundary conditions with absorbing boundary layers. The question of well posedness has been investigated in detail for various systems of equations and boundary conditions by Oliger and Sundström (1978).

Here we shall use a simple open boundary condition which consists of specifying the quantities (i.e., the Riemann invariants) that propagate along characteristics into the model domain and not specifying the quantities which propagate out. We refer to this as the *characteristic boundary condition*; a simple derivation appears in Fulton (1984). For a linearized version of the shallow water equations (2.1), one specifies $u'_\perp - \phi'/c$ everywhere on the boundary and $u'_\parallel$ wherever there is basic state inflow. Here $u_\perp$ and $u_\parallel$ denote the velocity components normal (positive outward) and parallel to the boundary, respectively, primes denote deviations from the basic state, and $c = \bar{\phi}^{1/2}$. For the full nonlinear Eqs. (2.1) or (2.2), one can instead specify $u_\perp - 2(\bar{\phi} + \phi)^{1/2}$ everywhere on the boundary and $u_\parallel$ on inflow; this effectively replaces the propagation speed $c = \bar{\phi}^{1/2}$ with $(\bar{\phi} + \phi)^{1/2}$ (Oliger and Sundström, 1978).

While the characteristic boundary condition yields a well-posed problem, it will not yield a good approximation unless the incoming quantities are specified correctly for the physical situation being modeled. The key here is the observation that each of the quantities specified in fact has contributions from both the balanced flow and the fast-propagating gravity waves; one wants to eliminate the incoming gravity wave component (assuming there are no sources of gravity waves outside the computational domain) without distorting the balanced flow. Simply setting the incoming quan-

tities to zero (the *homogeneous* form of the condition) may result in significant errors when there is balanced flow near the boundary. In general, one should set the incoming quantities approximately equal to what their contributions would be from the balanced flow on an infinite domain (the *inhomogeneous* form of the condition); if these specified values are good enough approximations then the incoming gravity wave components will be small.

Two other boundary conditions will also be considered in this study for comparison. The *balance* boundary condition is based on the assumption that near the boundaries the solution behaves like that of the corresponding balanced model. For a linearized, y-independent version of the shallow water model on an f-plane this condition takes the form

$$\left.\begin{array}{ll} \dfrac{\partial u'}{\partial x}-\dfrac{f}{c}u'=0 & \text{at}\quad x=x_a \\[2mm] \dfrac{\partial u'}{\partial x}+\dfrac{f}{c}u'=0 & \text{at}\quad x=x_b \end{array}\right\}, \qquad (2.3)$$

where $x_a \le x \le x_b$ is the one-dimensional model domain (Fulton, 1984). As before, v' must be specified wherever there is basic state inflow. Since no useful generalization of the balance condition to two dimensions in Cartesian coordinates is known, we will use this condition only in one-dimensional examples. The *wall* boundary condition simply sets $u_\perp = 0$ on the boundary; this yields a well-posed problem for the shallow water model (Oliger and Sundström, 1978).

The performance of the boundary conditions described above is measured in part by their ability to transmit gravity waves. In the linear model on an f-plane it can be shown that a gravity wave with unit amplitude incident on the boundary produces a reflected wave with amplitude R satisfying

$$|R|^2 = \frac{(\nu\cos\theta - \rho c)^2 + f^2\sin^2\theta}{(\nu\cos\theta + \rho c)^2 + f^2\sin^2\theta}, \qquad (2.4)$$

where θ is the angle between the outward normal to the boundary and the incident wavenumber vector $\mathbf{k}$, $\rho = |\mathbf{k}|$, and $\nu = (f^2 + \rho^2 c^2)^{1/2}$ is the gravity wave frequency. Figure 1 shows $|R|$ as a function of the dimensionless wavenumber $\rho c/f$ and the angle of incidence θ. Lower reflectivities can be achieved (e.g., Engquist and Majda, 1977) at the cost of increasingly complicated boundary conditions. In contrast, the balance and wall boundary conditions both have unit reflectivity and hence trap all gravity waves within the model domain. However, the balance condition is the superior of the two in the sense that it allows mass flow u through the boundary and does not distort the geostrophic component of the flow.

With perfectly reflecting (or periodic) boundary conditions, one can use the normal modes of the continuous model as basis functions for a spectral method
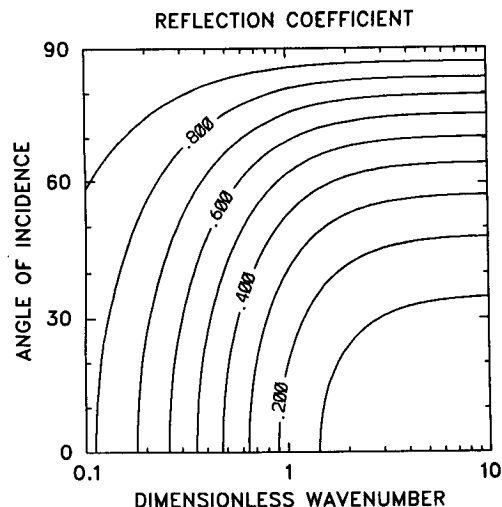

FIG. 1. Reflection coefficient $|R|$ for the characteristic boundary condition as a function of the dimensionless wavenumber $\rho c/f$ and the angle of incidence θ in degrees.

(e.g., Schubert and DeMaria, 1985). This normal mode approach represents the dynamics directly in the discretization, thus simplifying the problems of initialization and time differencing. However, in the general case where such boundary conditions are physically inappropriate this normal mode approach is not available; to obtain the high accuracy of a spectral method one can use a basis of Chebyshev polynomials as discussed in Part I. In the next section we formulate Chebyshev spectral models based on the shallow water equations with the boundary conditions discussed above.

3. Chebyshev spectral discretizations

Chebyshev spectral discretizations of the shallow water model on the domain $x_a \le x \le x_b$, $y_a \le y \le y_b$ are based on the expansion

$$\begin{bmatrix} u(x,y,t) \\ v(x,y,t) \\ \phi(x,y,t) \end{bmatrix} \approx \begin{bmatrix} u_{MN}(x,y,t) \\ v_{MN}(x,y,t) \\ \phi_{MN}(x,y,t) \end{bmatrix}$$
$$= \sum_{m=0}^{M}\sum_{n=0}^{N} \begin{bmatrix} \hat{u}_{mn}(t) \\ \hat{v}_{mn}(t) \\ \hat{\phi}_{mn}(t) \end{bmatrix} T_m(x')T_n(y'). \quad (3.1)$$

Here T_n is the Chebyshev polynomial of degree n, M and N are the spectral truncations in x and y, respectively, $x' = 2(x - x_a)/(x_b - x_a) - 1$, $y' = 2(y - y_a)/(y_b - y_a) - 1$, and carets denote spectral coefficients. The two methods described below differ in how they determine the spectral coefficients.

a. The Chebyshev–tau method

The tau approximation is obtained as described in section 2b of Part I, using an inner product which is a

double integral with the form (3.7) of Part I in both x' and y'. Starting from the advective form (2.1), the tau equations are

$$\frac{d\hat{u}_{mn}}{dt} + \hat{A}_{mn} - f\hat{v}_{mn} + \hat{\phi}_{mn}^{(1,0)} = \hat{F}_{mn}$$

$$\frac{d\hat{v}_{mn}}{dt} + \hat{B}_{mn} + f\hat{u}_{mn} + \hat{\phi}_{mn}^{(0,1)} = \hat{G}_{mn} \Bigg\} , \quad (3.2)$$

$$\frac{d\hat{\phi}_{mn}}{dt} + \hat{C}_{mn} + \hat{D}_{mn} + \bar{\phi}(\hat{u}_{mn}^{(1,0)} + \hat{v}_{mn}^{(0,1)}) = \hat{Q}_{mn}$$

where $\hat{A}_{mn}$, $\hat{B}_{mn}$, $\hat{C}_{mn}$ and $\hat{D}_{mn}$ are the spectral coefficients of the nonlinear terms $A = u\partial u/\partial x + v\partial u/\partial y$, $B = u\partial v/\partial x + v\partial v/\partial y$, $C = u\partial\phi/\partial x + v\partial\phi/\partial y$ and $D = \phi(\partial u/\partial x + \partial v/\partial y)$, respectively, and the superscripts $(1, 0)$ and $(0, 1)$ denote the spectral coefficients of x and y derivatives, respectively. Note that to take such derivatives in spectral space the standard Chebyshev derivative relation must be modified to take into account the domain length, e.g.,

$$c_{m-1}\hat{u}_{m-1,n}^{(1,0)} - \hat{u}_{m+1,n}^{(1,0)} = 2m\left(\frac{2}{x_b - x_a}\right)\hat{u}_{mn}, \quad (3.3)$$

where $c_0 = 2$ and $c_m = 1$ for $m > 0$. Although for simplicity we have assumed that f is constant in (3.2), the β-plane case also can be handled easily by noting that

$$(\widehat{yu})_{mn} = \frac{1}{2}(y_a + y_b)\hat{v}_{mn}$$

$$+ \frac{1}{4}(y_b - y_a)(c_{n-1}\hat{v}_{m,n-1} + \hat{v}_{m,n+1}), \quad (3.4)$$

where we take $\hat{v}_{m,N+1} = 0$, with a similar formula for $(\widehat{yv})_{mn}$.

As in most spectral models, one can compute the spectral coefficients of the nonlinear terms in (3.2) from the dependent variables $\hat{u}_{mn}$, $\hat{v}_{mn}$ and $\hat{\phi}_{mn}$ by the transform method of Eliasen et al. (1970) and Orszag (1970). For example, to compute the spectral coefficients of $u\partial u/\partial x$, one starts with $\hat{u}_{mn}$ and $\hat{u}_{mn}^{(1,0)}$ in spectral space, transforms these to physical space, multiplies them pointwise there, and transforms their product back to spectral space. The transforms can be computed via the standard discrete Chebyshev transform formulas [(3.17), (3.18) of Part I]; with at least $3M/2$ and $3N/2$ physical space points in x and y, respectively, all of the coefficients needed in the model are computed exactly, i.e., without aliasing. The transform method requires $O[MN(\log M + \log N)]$ operations using the FFT algorithm in the transforms and $O[MN(M + N)]$ operations without it, in contrast to the $O(M^2N^2)$ operations required when using the method of interaction coefficients.

The tau equations (3.2) are applied for most of the modes in the model, and the remaining degrees of free-

dom are used to satisfy the boundary conditions applied to the series expansion (3.1) as a whole. In practice, one can simply predict all model variables from (3.2), and then overwrite some of the last spectral coefficients in m or n as appropriate by new values obtained from the boundary conditions. For example, applying the wall condition at $x = x_a$ and transforming in y gives

$$\sum_{m=0}^{M} (-1)^m \hat{u}_{mn} = 0 \quad (0 \le n \le N), \quad (3.5)$$

from which we can diagnose $\hat{u}_{Mn}$ in terms of the other coefficients $\hat{u}_{mn}$ for each n. Other boundary conditions can be treated similarly, i.e., by applying them to the series as a whole. Note, however, that while linear boundary conditions can be treated easily in spectral space by transforming them in one direction as in (3.5), nonlinear boundary conditions must in general be treated directly in physical space, which is more awkward since it involves additional transforms.

The tau equations (3.2) were developed from the advective form (2.1), but the rotational form (2.2) can also be used. In practice, the rotational form is more efficient since it requires fewer transforms to compute the nonlinear terms than does the advective form (9 instead of 12). In either case the Chebyshev–tau equations closely parallel the continuous form of the model, and programming is relatively easy (with explicit time differencing) once a set of routines for standard operations such as transforms and derivatives is developed (a package of such routines is available from the authors). Nevertheless, Chebyshev–collocation methods are still easier, as will be seen below.

b. The Chebyshev-collocation method

The formulation of the Chebyshev–collocation version of the shallow water model is simple. We again use the expansion (3.1), and introduce the collocation points $(\bar{x}_j, \bar{y}_k)$ corresponding to $\bar{x}_j' = \cos(j\pi/M)$, $j = 0$, $\cdots$, M and $y_k' = \cos(k\pi/N)$, $k = 0$, $\cdots$, N. The collocation equations can then be written directly from (2.1) as

$$\frac{d\bar{u}_{jk}}{dt} + \bar{u}_{jk}\bar{u}_{jk}^{(1,0)} + \bar{v}_{jk}\bar{u}_{jk}^{(0,1)} - f\bar{v}_{jk} + \bar{\phi}_{jk}^{(1,0)} = \bar{F}_{jk}$$

$$\frac{d\bar{v}_{jk}}{dt} + \bar{u}_{jk}\bar{v}_{jk}^{(1,0)} + \bar{v}_{jk}\bar{v}_{jk}^{(0,1)} + f\bar{u}_{jk} + \bar{\phi}_{jk}^{(0,1)} = \bar{G}_{jk} \Bigg\}, \quad (3.6)$$

$$\frac{d\bar{\phi}_{jk}}{dt} + \bar{u}_{jk}\bar{\phi}_{jk}^{(1,0)} + \bar{v}_{jk}\bar{\phi}_{jk}^{(0,1)}$$

$$+ (\bar{\phi} + \bar{\phi}_{jk})(\bar{u}_{jk}^{(1,0)} + \bar{v}_{jk}^{(0,1)}) = \bar{Q}_{jk}$$

where a subscript jk denotes a value at the point $(\bar{x}_j, \bar{y}_k)$, and the superscripts $(1, 0)$ and $(0, 1)$ denote the x and y derivatives, respectively, as before. To compute these derivatives one uses the usual collocation procedure described in Part I: transform to spectral space,

take the derivative there, and transform back to physical space. A total of 12 transforms is required for (3.6), the same as for the corresponding collocation equations generated from the rotational form (2.2); these transforms are all one-dimensional and equal in length to the truncation, whereas the transforms needed in the tau method are all two-dimensional and longer than the truncation. No special treatment is needed for the nonlinear terms; other problems with more complicated nonlinearities (even transcendental ones) are handled with the same simplicity.

Boundary conditions for the collocation method are formulated exactly as in the continuous case, since the model variables $\bar{u}_{jk}$, $\bar{v}_{jk}$ and $\bar{\phi}_{jk}$ are carried at the boundary points. For example, to apply the wall condition one simply sets the corresponding values of $\bar{u}_{jk}$ and $\bar{v}_{jk}$ to zero, e.g., $\bar{u}_{Mk} = 0$, $k = 0, \cdots, N$, for the boundary $x = x_a$. Similarly, the characteristic boundary condition specifies the incoming combination $u_{\perp} - 2(\bar{\phi} + \phi)^{1/2}$ at the boundary; to determine both $u_{\perp}$ and ϕ there we simply require that the outgoing quantity $u_{\perp} + 2(\bar{\phi} + \phi)^{1/2}$ be unchanged from its value as predicted by the interior equations (3.6). However, the balance boundary condition (2.3) leads to implicit equations due to the global dependence of the Chebyshev derivative relation; this boundary condition is easier to implement in a predictive form, obtained by substituting for $\partial u/\partial x$ from the continuity equation.

The price one pays for the simplicity of the collocation method is the introduction of aliasing. In the nonlinear advection process there is a transfer of energy between scales of motion, with information from the resolvable scales [here, $T_m(x')T_n(y')$ for $m \leq M$ and $n \leq N$] cascading to the unresolvable scales ($m > M$ or $n > N$). In the tau method the contributions to the resolvable scales are computed exactly and the contributions to the unresolvable scales are simply neglected. In the collocation method the evaluation of a nonlinear term in general takes the information which should go into unresolvable scales and spreads it over the resolvable scales. The consequences of such aliasing in a Chebyshev-collocation model are not immediately clear. Orszag (1971, 1972) has made a case that aliasing need not be regarded as a type of error, and Fox and

Orszag (1973) suggest that the nonlinear numerical instabilities sometimes encountered in finite difference schemes (Phillips, 1959) may be avoided in collocation methods by using rotational forms of the equations such as (2.2). We will examine the latter point in the following section, in which we present results from the Chebyshev models developed above.

4. Numerical results

In this section we first examine the stability and accuracy of the Chebyshev spectral models developed above, using a simplified one-dimensional case for which analytical results can be computed. We then present numerical results from two-dimensional models and make some observations regarding the relative efficiencies of the tau and collocation methods.

a. One-dimensional results

Consider a y-independent version of the shallow water equations on an f-plane, linearized about a basic state at rest. In this simple case, the stability of the Chebyshev–tau and Chebyshev–collocation models with various (linear) boundary conditions can be analyzed by matrix methods. We do this here by computing numerically the eigenvalues of the space discretized model for various values of model parameters. Table 1 gives the largest time step Δt for which the various models are absolutely stable using the six explicit time differencing schemes considered in Part I. Additionally, results are given for the filtered leapfrog scheme (FLF), i.e., the leapfrog scheme coupled with the Asselin–Robert filter (Haltiner and Williams, 1980), using the filter parameter $\gamma = 0.01$ for the wall and balance boundary conditions and $\gamma = 0.5$ for the characteristic boundary conditions. The time steps are normalized by the speed c, the spectral truncation N, and the domain size $l = x_b - x_a$, thus making them essentially independent of these parameters; they are also normalized by the number s of "stages" (i.e., function evaluations per time step) to indicate relative efficiency.

With reflecting (wall or balance) boundary conditions, all eigenvalues are pure imaginary, and the FLF scheme is the most efficient; however, the RK4 scheme

TABLE 1. Normalized time step $cN^2\Delta t/(sl)$ for stability for the y-independent shallow water model with various time differencing schemes and boundary conditions.

Scheme	Chebyshev–collocation			Chebyshev–tau		
	Wall	Balance	Characteristic	Wall	Balance	Characteristic
FOR	0.00	0.00	2.20	0.00	0.00	1.70
MAT	1.12	1.11	3.41	0.45	0.45	0.42
AB2	0.00	0.00	4.05	0.00	0.00	0.85
RK2	0.00	0.00	3.72	0.00	0.00	0.85
AB4	0.96	0.95	2.02	0.39	0.38	0.25
RK4	1.59	1.57	3.84	0.64	0.63	0.59
FLF	2.22	2.20	3.25	0.89	0.89	0.56

may be a better choice in general, since it is much more accurate (and has no computational modes) and is only slightly less efficient. With characteristic boundary conditions, some eigenvalues have negative real parts (corresponding to energy propagating out of the domain), and the normalized time step required for stability is very close to that obtained in Part I for the one-dimensional advection equation (taking into account the domain length of the latter problem). In particular, the FLF scheme is unstable except for very small time steps or very large filter parameters (e.g., γ = 0.5), the RK4 scheme is probably the best choice for the collocation method, and the RK2 and AB2 schemes are slightly more efficient for the tau method.

In the linearized, y-independent problem described above, the continuous solution on an infinite domain converges to that of the corresponding balanced (geostrophic) model as $t \to \infty$ (provided the initial conditions are geostrophic and the forcing tends to zero as $t \to \infty$). The latter solution (the final adjusted state) can be computed easily from the potential vorticity equation, thus providing a standard to which the numerical solution computed on a bounded domain can be compared. To test the accuracy of the models with the various boundary conditions, we consider a test case with the initial fields u', v', ϕ' all zero and the geopotential forced by

$$Q(x, t) = -\Phi e^{-(x/x_0)^2} 4t^2 t_0^{-3} e^{-2t/t_0}, \qquad (4.1)$$

where x_0 is the e-folding width. The time dependence of (4.1) is such that the forcing starts at zero, rises smoothly to a peak at $t = t_0$, and decays smoothly to zero, with the time integral of the forcing (and hence the final adjusted state on an infinite domain) independent of the time scale t_0. For the results presented here we use x_0 = 200 km and $f = 5 \times 10^{-5}$ s^{-1} (corresponding to the latitude 20°N). Figure 2 shows the analytical final adjusted state on an infinite domain for c = 250, 50 and 10 m s^{-1}, corresponding roughly to the external mode, first internal mode, and sixth internal mode in a compressible hydrostatic atmosphere, respectively. The forcing amplitude (arbitrary in this linear problem) was chosen as $\Phi = c^2/10$ so that in each case the final fields u', v' and ϕ'/c would be of order 1 m s^{-1}.

In view of the symmetry of the forcing (4.1), the numerical models were run on the bounded domain $[x_a, x_b]$ = [0, 1000 km] with a wall at x = 0. The spectral truncation N = 16 was sufficient to resolve the solution, and RK4 time differencing was used with the time step chosen small enough that time differencing errors were negligible. Boundary values needed for the inhomogeneous characteristic condition were obtained from the solution of the corresponding balanced model. Two forcing time scales were considered: "slow" forcing with t_0 = 12 h and "fast" forcing with t_0 = 3 h, with the fast forcing generating more gravity waves. Table 2 gives the L_2 errors (computed with a trapezoidal ap-

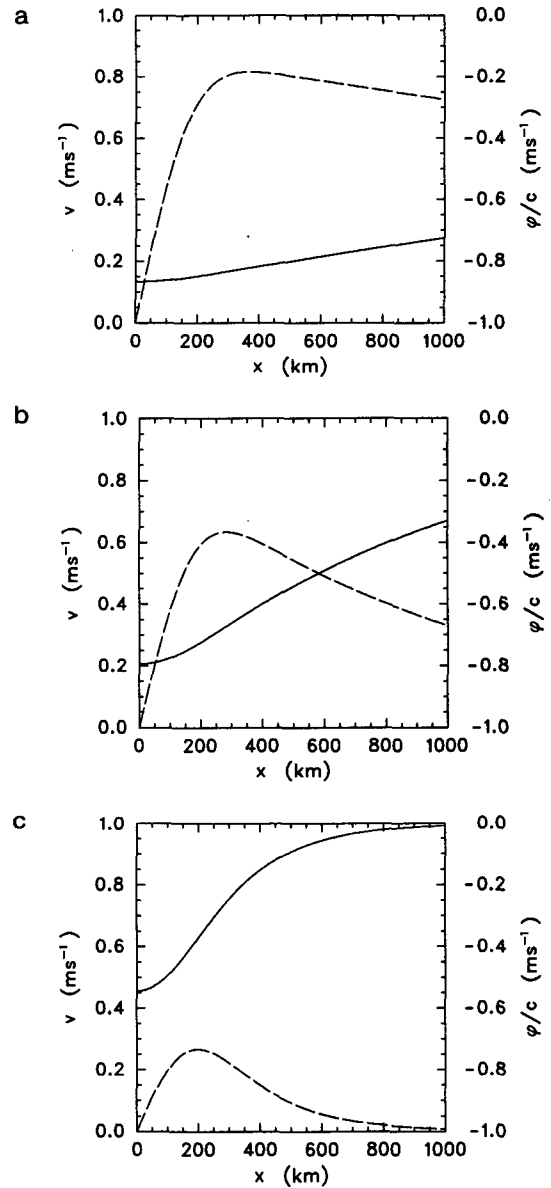FIG. 2. Analytical final state of the linearized y-independent shallow water model for (a) c = 250 m s^{-1}, (b) c = 50 m s^{-1}, and (c) c = 10 m s^{-1}. The dashed curve represents v (left scale) and the solid curve represents ϕ/c (right scale).

proximation as in (3.23) of Part I and scaled by the domain length) in the Chebyshev–tau solution at t = 5 days (the errors were averaged from t = 4 days to t = 5 days where necessary to smooth the effects of trapped gravity waves). From these results we conclude the following. First, the inhomogeneous characteristic boundary condition gives the best results in each case, usually by many orders of magnitude. Second, with little balanced flow near the boundary (c = 10 m s^{-1}) the two forms of the characteristic condition give comparable errors (note that c = 10 m s^{-1} is small enough

TABLE 2. Errors in the Chebyshev–tau solution of the y-independent shallow water model with a wall at x_a and various boundary conditions at x_b, (I and H denote the inhomogeneous and homogeneous forms of the characteristic (C.) boundary condition, and numbers in parentheses denote exponents of ten).

c (m s^{-1})	t_0 (hours)	Boundary condition	L_2 errors (m s^{-1}) at $t = 120$ hours		
			u	v	ϕ/c
250	12	I.C.	3.4 (−9)	1.0 (−6)	2.8 (−8)
		H.C.	6.8 (−8)	8.6 (−2)	7.4 (−1)
		Balance	5.1 (−3)	6.1 (−4)	4.8 (−3)
		Wall	6.8 (−4)	4.4 (−1)	3.7 (0)
	3	I.C.	4.3 (−14)	1.0 (−6)	1.3 (−8)
		H.C.	8.3 (−15)	8.6 (−2)	7.4 (−1)
		Balance	2.4 (−1)	2.7 (−2)	2.1 (−1)
		Wall	4.1 (−2)	4.4 (−1)	3.7 (0)
50	12	I.C.	5.0 (−8)	1.0 (−6)	9.6 (−8)
		H.C.	1.4 (−7)	1.4 (−1)	2.6 (−1)
		Balance	3.6 (−2)	1.5 (−2)	2.6 (−2)
		Wall	1.4 (−2)	1.9 (−1)	3.5 (−1)
	3	I.C.	2.3 (−14)	1.0 (−6)	6.0 (−8)
		H.C.	1.0 (−13)	1.4 (−1)	2.6 (−1)
		Balance	4.2 (−1)	1.9 (−1)	3.5 (−1)
		Wall	3.0 (−1)	2.1 (−1)	4.7 (−1)
10	12	I.C.	7.4 (−3)	6.8 (−3)	4.5 (−3)
		H.C.	7.3 (−3)	6.8 (−3)	5.1 (−3)
		Balance	2.3 (−2)	1.7 (−2)	1.3 (−2)
		Wall	2.1 (−2)	2.0 (−2)	1.6 (−2)
	3	I.C.	3.0 (−2)	2.2 (−2)	1.5 (−2)
		H.C.	3.0 (−2)	2.2 (−2)	1.5 (−2)
		Balance	1.1 (−1)	7.8 (−2)	7.4 (−2)
		Wall	1.2 (−1)	7.3 (−2)	8.0 (−2)

that here the gravity–inertia waves are substantially reflected, and the errors decay only slowly with time). Third, the balance condition is preferable to both the wall and homogeneous characteristic conditions when there is significant balanced flow near the boundary (c = 50 and 250 m s^{-1}) and the forcing is slow. Finally, with fast forcing the balance and wall conditions give comparably poor results.

b. Two-dimensional results

To test the Chebyshev spectral versions of the two-dimensional shallow water model we use a two-dimensional analogue of the above test case, forcing the geopotential field by the mass sink

$$Q(x, y, t) = \Phi \exp\left[-\left(\frac{x - x_c}{x_0}\right)^2 - \left(\frac{y - y_c}{y_0}\right)^2 \right] 4t^2 t_0^{-3} e^{-2t/t_0},$$

(4.2)

with e-folding width $x_0 = y_0 = 200$ km, time scale t_0 = 6 hours, and amplitude $\Phi = 6250$ m^2 s^{-2}. For all of the results presented below the model domain is

[−2000 km, 2000 km] × [−2000 km, 2000 km], centered at 30°N, and $c = 50$ m s^{-1}. We use the spectral truncation $M = N = 24$; note that in a finite difference model with the same number of degrees of freedom, the e-folding width of the forcing would be approximately equal to the mesh spacing.

First, we examine the performance of the characteristic boundary condition by running the linear model on an f-plane with the initial u, v and ϕ fields all zero and the forcing centered at $x_c = y_c = 0$. Figure 3 shows the Chebyshev–tau solution at $t = 6$, 12 and 24 h. In these and subsequent figures, the geopotential field ϕ/c is represented by contour lines (with contour interval 2 m s^{-1}) and the velocity field u, v by vectors as discrete points (scaled so a vector from one point to the next would have magnitude 15 m s^{-1}). In this experiment the gravity waves generated by the forcing propagate out of the domain with no evidence of reflection, leaving behind a geostrophically balanced vortex. Chebyshev–collocation results for this test case are very similar and hence are not shown here.

For a more realistic test case we initialize the model with the zonal flow

$$u(x, y, 0) = -U \cos\left[\pi\left(\frac{y - y_a}{y_b - y_a}\right) \right]$$

(4.3)

with $U = 7.5$ m s^{-1} and ϕ in geostrophic balance on a β-plane, and then use the forcing (4.2) with center located initially at $x_c = 1000$ km, $y_c = -1000$ km and advected with the initial flow (4.3) to generate a vortex in the region of easterly flow. The initial u, v and ϕ are used to provide the boundary values required by the inhomogeneous form of the characteristic boundary condition as the solution evolves. We ran the nonlinear Chebyshev–collocation version of the model for this test case using explicit RK4 time differencing with a time step of 10 minutes; on a CRAY-1 computer the execution time required was approximately 7 seconds per model day. Figure 4 shows the resulting solution at $t = 1, 2, 4, 6$ and 8 days. After one day the forcing is essentially zero, and the vortex simply propagates with the surrounding flow, recurving due to the effects of β and the vorticity gradient of the surrounding flow.

The results in Fig. 4 were computed by the collocation method using the advective form (2.1). Corresponding solutions at $t = 1$ day computed by the collocation method using the rotational form and by the tau method using both forms are shown in Fig. 5. The collocation solution from the rotational form differs markedly from the other three solutions, indicating that the advective form is probably the most appropriate for the collocation method. In contrast, the tau method gives very similar results with either form of the equations (although the rotational form is more efficient, as explained below). We note that even though the tau and collocation methods treat the nonlinear terms differently, the solutions produced are quite similar. In
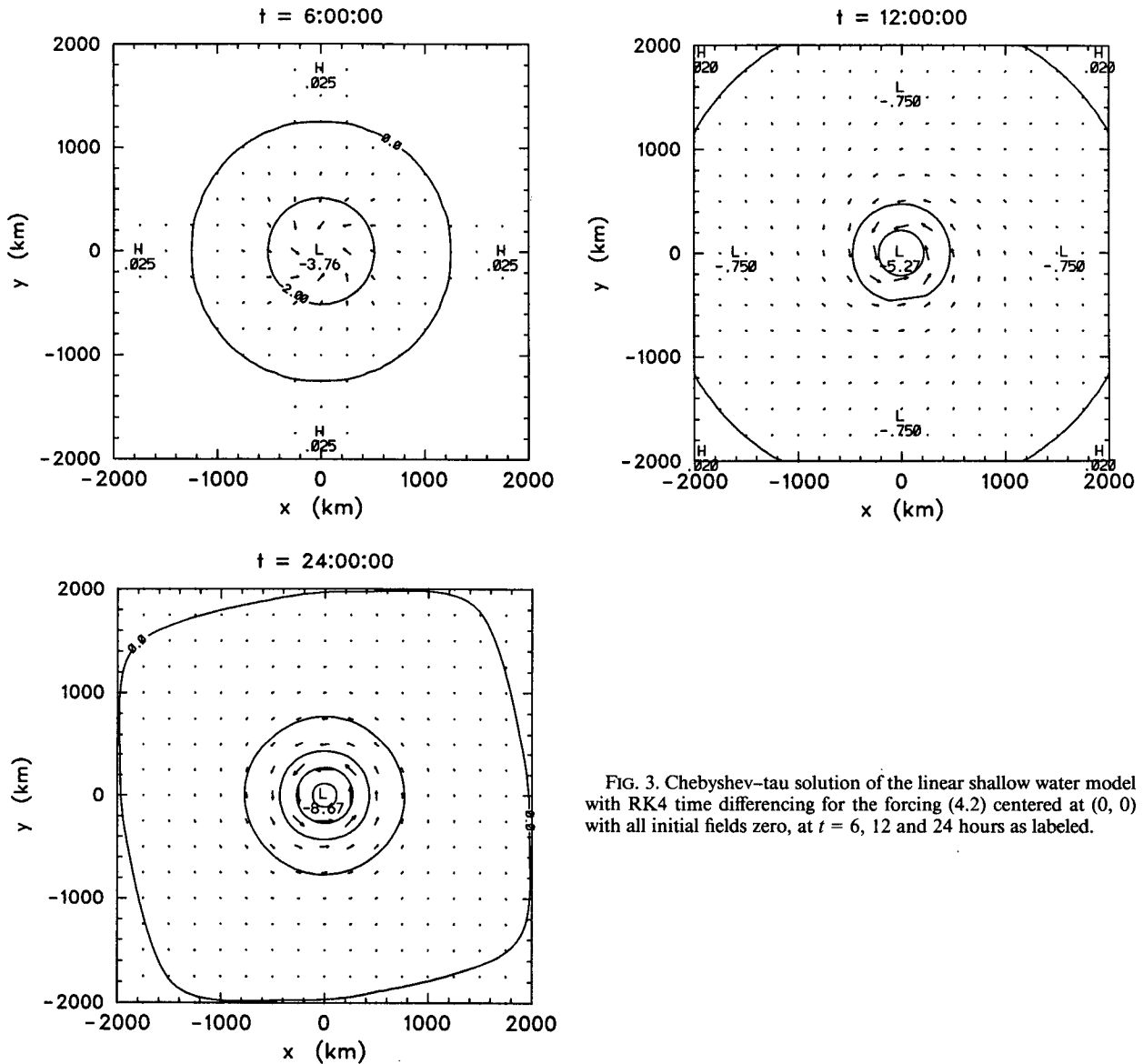
FIG. 3. Chebyshev–tau solution of the linear shallow water model with RK4 time differencing for the forcing (4.2) centered at (0, 0) with all initial fields zero, at t = 6, 12 and 24 hours as labeled.

particular, the collocation method has aliasing errors, and yet we saw no evidence of nonlinear numerical instability in the experiments which we performed. These conclusions may not extend, however, to physical situations in which the distribution of energy between spatial scales is different.

All of the numerical results for this test case show some small oscillations in the geopotential field. One possible source is the cascade of energy from large to small scales due to nonlinear interactions, coupled with the limited resolution of the numerical model: features which are not adequately resolved in spectral methods tend to cause oscillations across the domain on the smallest resolvable scales. Alternately, they could also be due to inward propagation of errors due to inexact specification of boundary data. In either case, simply

using more Chebyshev modes will not completely eliminate the problem. Rather, a small amount of dissipation should be used, both to represent the physical process of energy cascade to scales which are not resolved, and to smooth errors from inexact boundary data. Note, however, that no dissipation is required simply to run the model or to cover up inadequacies of the numerical method: the Chebyshev models run quite well without any dissipation, even in the above case where the number of modes used is barely sufficient to resolve the solution.

One significant difference between the tau and collocation methods for this problem is their efficiency. Table 3 shows the computer time required to evaluate all terms in the model equations using each of the various discretization methods and equation forms with
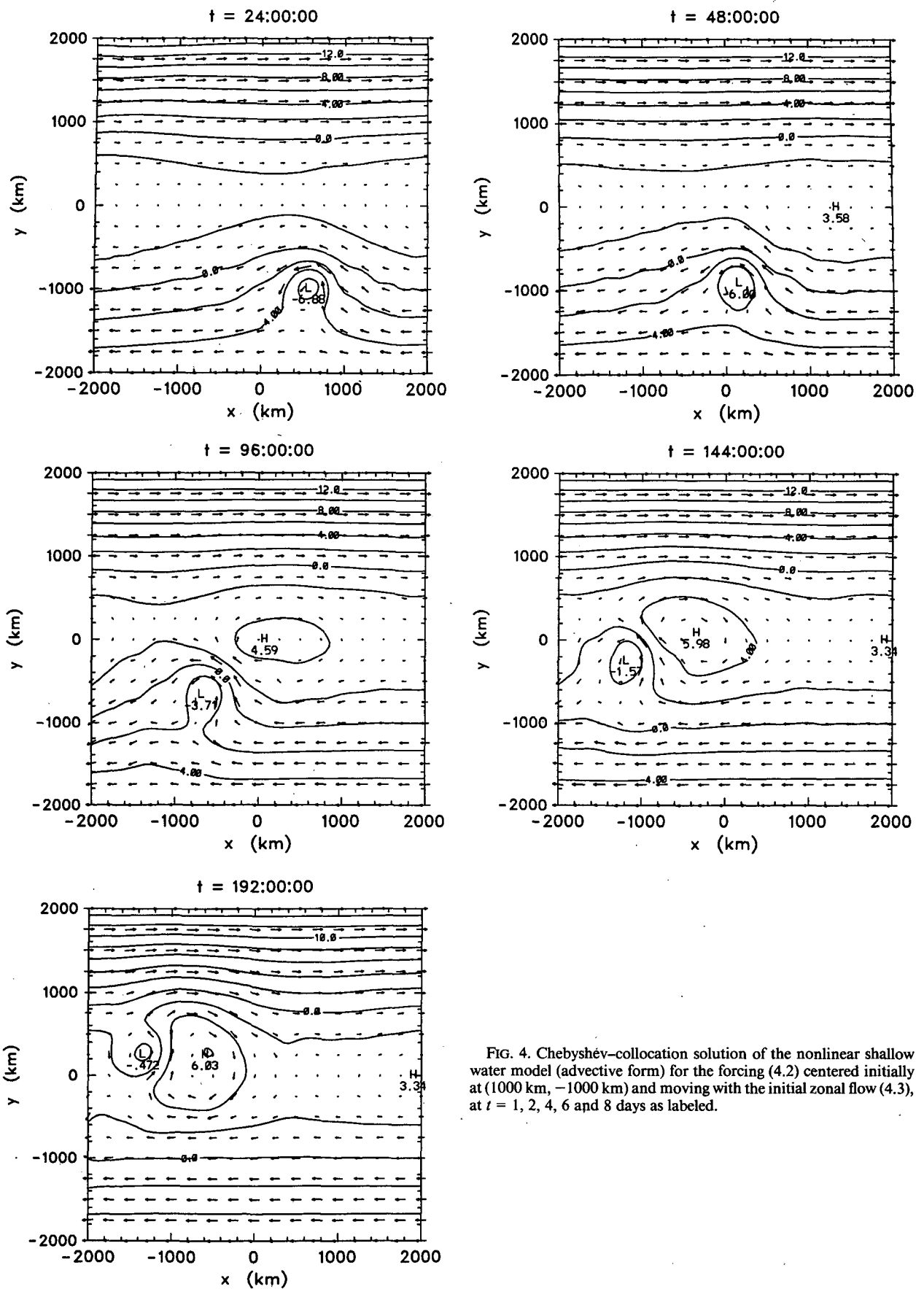
t = 24:00:00

t = 48:00:00

t = 96:00:00

t = 144:00:00

t = 192:00:00

FIG. 4. Chebyshév–collocation solution of the nonlinear shallow water model (advective form) for the forcing (4.2) centered initially at (1000 km, −1000 km) and moving with the initial zonal flow (4.3), at $t = 1, 2, 4, 6$ and 8 days as labeled.

a   t = 24:00:00

b   t = 24:00:00
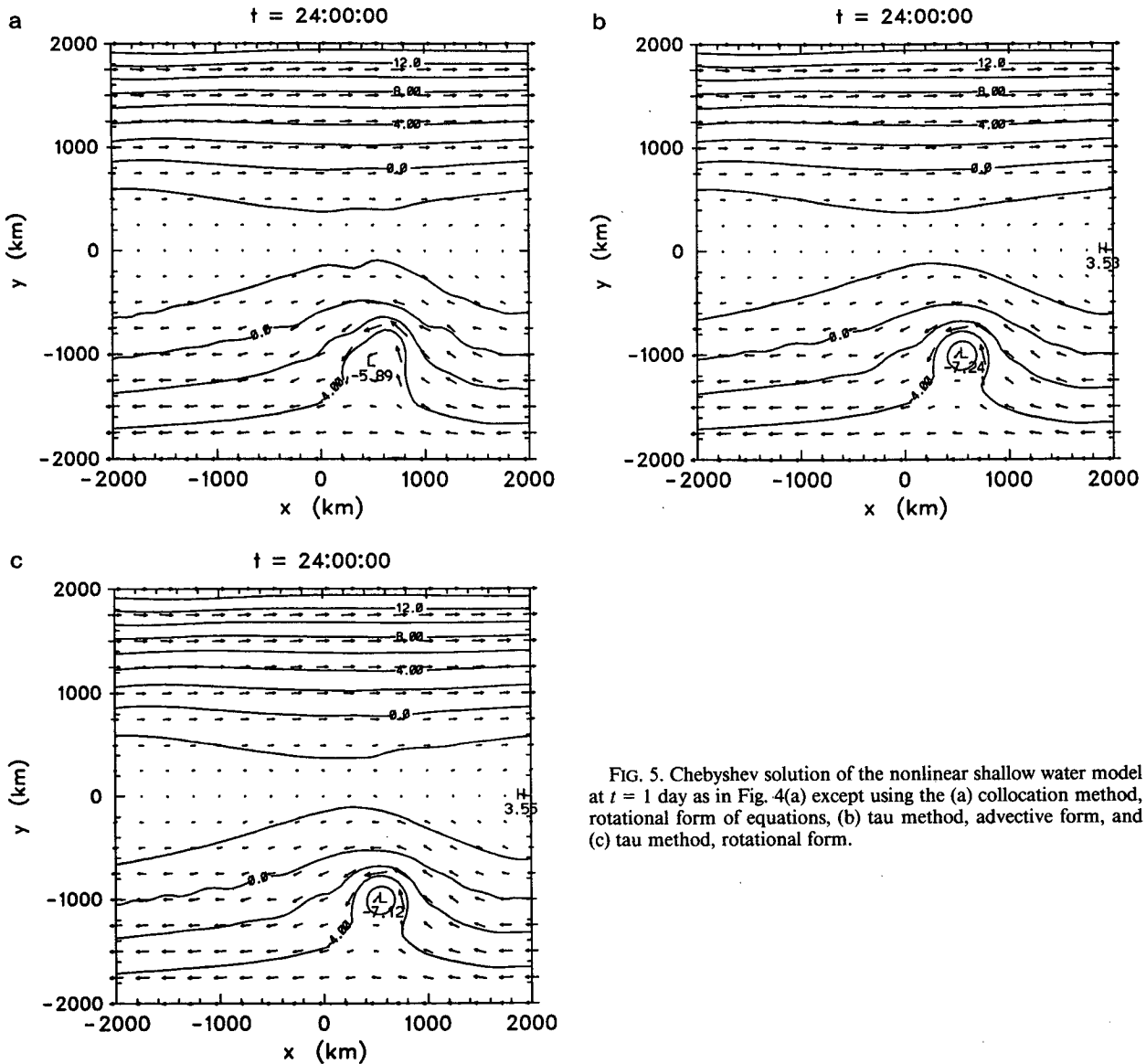

c   t = 24:00:00

FIG. 5. Chebyshev solution of the nonlinear shallow water model at $t = 1$ day as in Fig. 4(a) except using the (a) collocation method, rotational form of equations, (b) tau method, advective form, and (c) tau method, rotational form.

the spectral truncation $M = N = 24$; these estimates are based on timing tests performed on a CRAY-1 computer using fully vectorized code. In the nonlinear case the collocation method requires less work per time step than the tau method by a factor of about 3, primarily because the transforms involved are shorter in length and one-dimensional. In addition, with explicit

TABLE 3. CRAY-1 execution time (ms) required to evaluate all terms in the shallow water model with $M = N = 24$.

Equation form	Chebyshev–collocation	Chebyshev–tau
Linear	2.8	1.9
Advective	3.9	14.2
Rotational	3.9	11.5

time differencing the collocation method allows time steps which are larger by a factor of about 5 (although a matrix stability analysis is impractical in two dimensions, numerical experience indicates that the stability conditions for the various models generally follow those of the one-dimensional models reported in Table 1). Thus while both the Chebyshev–tau and Chebyshev–collocation methods work, the collocation method is about an order of magnitude more efficient—with explicit time differencing—for the same number of modes.

5. Concluding remarks

In this paper we have applied Chebyshev spectral methods to the nonlinear shallow water equations in

two dimensions. The methods are straightforward to implement, and result in models which have the high accuracy of the spectral approach while allowing the use of open boundary conditions. While the test cases considered here do not involve enough physics to be regarded as models of the real atmosphere, the results presented suggest that the Chebyshev methods could be used in limited-area meteorological models to treat the dynamics with high accuracy.

Two basic projections were considered here: tau and collocation. Numerical results indicate that Chebyshev–tau models of the shallow water equations should be based on the rotational form of the equations (for efficiency), while Chebyshev–collocation models should be based on the advective form (for accuracy). The tau method may seem preferrable, since it involves no aliasing, but in practice the aliasing introduced by the collocation method may not be a problem: the numerical results from both methods were similar for the test cases studied here, and we found no evidence of nonlinear numerical instability. On the other hand, the collocation method is about an order of magnitude more efficient than the tau method, at least with explicit time differencing, and is also somewhat easier to program. Thus the collocation method is probably the more useful of the two. Given the close connection between the shallow water equations and the full primitive (hydostatic) equations, we believe that these conclusions should carry over to the latter system, taking into account, of course, the existence of vertical modes with widely differing phase speeds.

Several topics deserve further study. First, Chebyshev models should in practice include a small amount of dissipation, not to cover up for inadequacies of the numerical method, but to represent the physical cascade of energy to scales which are not resolved in the model. Unfortunately, simply including friction terms proportional to the Laplacian of model variables leads to a stability condition (for explicit time differencing) of the form $\Delta t = O(N^{-4})$; this is probably unduly restrictive when the specific form of the friction is simply a somewhat arbitrary representation of dissipation. One approach here is to treat such friction terms implicitly; another is to replace them altogether with a spectral filtering of the model coefficients. Second, while the Chebyshev models are practical with explicit time differencing, in cases where propagating gravity waves contribute little to the solution one might obtain substantial gains in efficiency (especially with the tau method) by treating the terms responsible for these waves implicitly, as is done in finite difference and global spectral models (e.g., Kwizak and Robert, 1971). Finally, the accuracy and efficiency of Chebyshev spectral and finite difference models of the shallow water equations should be compared in detail. We are presently studying all of these topics.

Finally, we note that the spectral techniques considered in this study may in fact be *too* accurate to be of much use in some limited-area models. As stated previously, one of the fundamental problems in limited-area modeling is to minimize the impact of computationally imposed boundaries. The numerical examples presented here suggest that errors due to inexact specification of boundary conditions and boundary values may be much larger than spectral discretization errors. In more complicated models, especially those intended for operational use in forecasting, additional sources of error include the specification of initial fields and the representation of small-scale physical processes such as those occurring in cumulus clouds. Chebyshev spectral methods probably allow such problems to be solved much more accurately than they can be specified at present. Spectral methods may effectively eliminate discretization errors from limited-area models; much more work is needed to reduce the other sources of error.

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