The Role of Gravity Waves in Slowly Varying in Time Tropospheric Motions near the Equator

G. L. BROWNING*

NOAA Forecast Systems Laboratory, Boulder, Colorado

H.-O. KREISS

Department of Mathematics, University of California, Los Angeles, Los Angeles, California

W. H. SCHUBERT

Colorado State University, Fort Collins, Colorado

(Manuscript received 22 March 1999, in final form 24 April 2000)

ABSTRACT

A mathematical theory was recently developed on the relationship between the dominant and gravity wave components of the slowly varying in time solutions (solutions varying on the advective timescale) corresponding to midlatitude mesoscale motions forced by cooling and heating. Here it will be shown that slowly varying in time equatorial motions of any length scale satisfy the same balance between the vertical velocity and heating as in the midlatitude mesoscale case. Thus any equatorial gravity waves that are generated will have the same time- and depth scales and the same size of pressure perturbations as the corresponding dominant component, a horizontal length scale an order of magnitude larger than that of the heat source, and an order of magnitude smaller velocity than the corresponding dominant component. In particular, in the large-scale equatorial case, when the heating has a timescale O(1 day), horizontally propagating gravity waves with a timescale O(1 day) and a length scale O(10 000 km) can be generated. But in the large-scale equatorial case when the heating has a timescale O(10 days), balanced pressure oscillations with a timescale O(10 days) are generated. It is also shown that if a solution of the diabatic system describing equatorial flows (and hence equatorial observational data in the presence of heating) is written in terms of a series of the modes of the linear adiabatic system for those flows, then a major portion of the dominant solution is projected onto gravity wave modes, and this result can explain the confusion over the relative importance of equatorial gravity waves.

1. Introduction

The system of equations used to describe fluid motions in the lower atmosphere is a time-dependent hyperbolic system with multiple timescales (Kreiss 1979, 1980). Thus there exist solutions in the lower atmosphere that essentially only vary on the advective timescale. These solutions can be obtained either by using the reduced system that is devoid of any fast waves or by the appropriate choice of initial conditions (a process called initialization) for the full system of equations. In the early days of meteorology, the quasigeostrophic system was used to describe large-scale midlatitude motions (Charney 1948; Phillips 1956) and contained only a single timescale. However, later the meteorological community began to use the primitive equations (Smagorinsky 1963), which have multiple timescales. When observational data are used as initial values for a model based on the primitive equations, large-amplitude, high-frequency gravity waves can be excited, and those waves can have a detrimental effect on the physical parameterizations in the model. In the large-scale case, it is well known that the removal of the gravity waves from the solution has little impact on the accuracy of the forecast. Thus to overcome the adverse impact of the gravity waves on the physical parameterizations in the large-scale case, a number of initialization schemes were developed, beginning with the geostrophic approximation and balance equation (e.g., Hinkelmann 1951; Charney 1955) and proceeding to increasingly more sophisticated methods such as the nonlinear nor-

* Additional affiliation: Cooperative Institute for Research in the Atmosphere, Colorado State University, Fort Collins, Colorado.

Corresponding author address: Dr. Gerald Browning, Cooperative Institute for Research in the Atmosphere, David Skaggs Research Center #3B906, 325 Broadway, Boulder, CO 80303.
E-mail: browning@fsl.noaa.gov

mal mode initialization method (Machenhauer 1977; Baer 1977; Baer and Tribbia 1977) and the bounded derivative method (Kreiss 1979, 1980).

Although the nonlinear normal mode initialization method has been applied successfully to the large-scale adiabatic case, there has been considerable confusion about the role of gravity waves in the initialization process in the presence of diabatic heating. The nonlinear normal mode initialization method is implemented in spectral space, and when heating is present, it is difficult to determine exactly which modes should be retained and what the corresponding balances in physical space should be (e.g., see discussion of this topic in Wergen 1988). Salby and Garcia (1987) and Garcia and Salby (1987) have used normal mode methods on the linearized primitive equations to study the transient response to localized episodic heating. Their solution procedure involves the separation of the problem into horizontal and vertical structure equations. The vertical structure problem involves the solution of an inhomogeneous second-order ordinary differential equation, with the inhomogeneous term originating from diabatic heating. Because of this form of the vertical structure equation, Salby and Garcia conclude that the dispersion relation can be usefully applied only above the heating. This situation arises because above the heating the behavior of the motion is determined by the homogeneous solution of the vertical structure equation, that is, there the vertical scale is related to frequency and wavenumber through the dispersion relation, but within the heating the vertical structure equation is inhomogeneous (there the solution includes a particular part that obeys no dispersion relationship). Thus, Salby and Garcia conclude that a discussion of dispersion characteristics, for example, energy in various normal modes, is problematic at the levels where diabatic heating is present. As a result, they restrict their discussion of the energy in different modes to the lower stratosphere, that is, above the diabatic heating. Because of the above problem, diabatic nonlinear normal mode balancing near the equator has been done in an ad hoc manner with no explanation of the relevant constraints in physical space (Wergen 1988). [Because of the complexities that arose in the diabatic implementation, diabatic nonlinear normal mode initialization has been mostly abandoned, and initialization has been incorporated through the use of penalty parameters in four-dimensional variational data assimilation (C. Temperton 1999, personal communication).] Also, there has been considerable controversy about the importance of gravity waves in smaller scales of motion (e.g., see discussion of this topic in the introduction of Browning and Kreiss 1997) and near the equator (e.g., see review article discussing 30–60-day equatorial gravity waves by Madden and Julian 1994).

Recently Browning and Kreiss (1997) have developed a mathematical theory that shows that a midlatitude mesoscale storm driven by cooling and heating consists of two parts that do not interact significantly with each other. The dominant component contains most of the energy of the solution in the neighborhood of the storm and is therefore the meteorologically significant part of the solution. The other part of the solution, the gravity wave component, consists of large-scale gravity waves that have the same time- and depth scales and the same amplitude of pressure perturbations as the dominant component. The gravity waves propagate horizontally away from the storm and can last for a considerable period of time after the storm has dissipated. This theory does not rely on the normal modes of the homogeneous system, but rather is based on a scaling of the equations in physical space, that is, on the standard bounded derivative theory (BDT) approach. Thus, unlike the normal mode approach, the new theory is applicable in the troposphere both inside and outside the region of diabatic heating. In this paper, it is shown that all slowly varying in time tropospheric motions near the equator satisfy the same balance between the vertical velocity and heating as in the midlatitude mesoscale case. Thus the midlatitude reduced system for mesoscale flows can be used to accurately describe all slowly varying in time tropospheric motions near the equator.

The outline of this paper is as follows. Section 2 contains the scaling of the three-dimensional dynamical equations for the set of slowly evolving tropospheric motions near the equator. Section 3 contains the parameters and corresponding scaled system for slowly evolving mesoscale flows and a discussion of the applicability of the new theory to the equatorial mesoscale case. Section 4 contains the scaled system for large-scale equatorial flows with a period $O(1\text{ day})$ and explains how the new gravity wave theory can be used to prove that the gravity waves that are generated can have a horizontal length scale $O(10\,000\text{ km})$. Then it is shown that when the period of the heating is $O(10\text{ days})$, balanced pressure waves with a period $O(10\text{ days})$ can be generated (cf. Madden and Julian 1994). In section 5 a simple example shows that an expansion of the diabatic solution into a series of normal modes of the adiabatic system can be misleading as to the amount of energy in gravity waves and how the simple version of nonlinear normal mode initialization (Machenhauer 1977) leads to an initialization constraint that has a $O(1)$ error when used to describe the dominant component of the solution. Several numerical examples that illustrate the theory are presented in section 6, and section 7 contains the conclusions.

2. Scaling for slowly evolving flows near the equator

The first step in the use of the BDT to understand slowly evolving flows in the atmosphere is to scale the differential equations that describe their evolution in time according to their properties in physical space (Browning et al. 1980). Here a scaling that describes the complete set of slowly evolving tropospheric flows near the equator will be derived in a manner similar to the midlatitude case (Browning and Kreiss 1986). The inviscid adiabatic Eulerian equations used to describe motions in the lower atmosphere are (e.g., Browning and Kreiss 1986)

$$\frac{ds}{dt} = 0, \tag{2.1a}$$

$$\frac{d\mathbf{V}}{dt} + \rho^{-1}\boldsymbol{\nabla}p + f(\mathbf{k} \times \mathbf{V}) + g\mathbf{k} = 0, \tag{2.1b}$$

$$\frac{dp}{dt} + \gamma p \boldsymbol{\nabla} \cdot \mathbf{V} = 0, \tag{2.1c}$$

where t is time, $s = \rho p^{-1/\gamma}$ is proportional to the reciprocal of the potential temperature, $\mathbf{V} = (u, \upsilon, w)^{\mathrm{T}}$ is velocity, ρ is density, and p is pressure. Also $\mathbf{k} = (0, 0, 1)^{\mathrm{T}}$ is the unit vector in the vertical direction, $g = 9.8$ m s^{-1} is the constant gravity acceleration, $\gamma = 1.4$ is the adiabatic exponent, and $d/dt = \partial/\partial t + u(\partial/\partial x) + \upsilon(\partial/\partial y) + w(\partial/\partial z)$. We assume that the Coriolis parameter f is given by the tangent plane approximation $f = 2\Omega[\sin\theta_0 + (y/r)\cos\theta_0]$, where $2\Omega \approx 10^{-4}$ s^{-1} is the earth's angular speed, $\theta_0 = 0$ is the latitude of the coordinate origin, and $r \approx 10^7$ m is the radius of the earth.

Now dimensionless variables will be introduced to identify the relative magnitudes of all terms in the equations. For the independent variables consider the change of variables

$$x = Lx', \qquad y = Ly', \qquad z = Dz', \qquad t = Tt', \tag{2.2}$$

where L, D, and T are the representative scales along the x (or y), z, and t axes, respectively. (To simplify the presentation, only the equal horizontal length-scale case is considered here.) The dependent variables are scaled according to the relations

$$u = Uu', \qquad \upsilon = U\upsilon', \qquad w = Ww',$$
$$g = 10g', \tag{2.3a}$$
$$p = P_0[p_0(z) + S_1 p'],$$
$$\rho = R_0[\rho_0(z) + S_1\rho'], \tag{2.3b}$$

where U and W are the representative velocity scales along the x (or y) and z axes, respectively; $P_0 = 10^5$ kg m^{-1} s^{-2} and $R_0 = 1$ kg m^{-3} are typical mean surface values of the pressure and density; and the variation of the mean with height is taken into account through the functions $p_0(z)$ and $\rho_0(z)$. The parameter S_1 $(0 < 10S_1 < 1)$ represents an assumption that the deviation of the pressure from its horizontal mean is small. Solutions that vary on the advective timescale are called slowly varying in time solutions. For these solutions it is assumed that the timescale is $T = L/U$ and that after the scaling all primed variables and a few of their time derivatives are $O(1)$.

Substituting (2.2) and (2.3) into (2.1) (see Browning and Kreiss 1986 for details), the scaled system describing the complete set of slowly evolving flows near the equator is

$$\frac{ds}{dt} - (10S_1)^{-1}S_2\tilde{s}w = 0, \tag{2.4a}$$

$$\frac{du}{dt} + S_3\rho_0^{-1}p_x - S_4 y\upsilon = 0, \tag{2.4b}$$

$$\frac{d\upsilon}{dt} + S_3\rho_0^{-1}p_y + S_4 yu = 0, \tag{2.4c}$$

$$\frac{dw}{dt} + S_1S_5(\rho_0^{-1}p_z + gs) = 0, \tag{2.4d}$$

$$\frac{dp}{dt} + S_1^{-1}\gamma p_0(u_x + \upsilon_y + S_2 w_z) = 0, \tag{2.4e}$$

where $\tilde{s} = 10D\,[\ln(\rho_0 p_0^{-1/\gamma})]_z$, the dimensionless parameters S_i $(i = 2, \ldots, 5)$ are

$$S_2 = D^{-1}TW, \qquad S_3 = S_1P_0(R_0U^2)^{-1},$$
$$S_4 = 2\Omega TL/r, \qquad S_5 = TP_0(DR_0W)^{-1}, \tag{2.5}$$

and for simplicity the prime notation and a number of terms that are not essential to understanding have been dropped.

3. Slowly evolving mesoscale motions near the equator

In a typical mesoscale convective system (MCS), there are two distinct regions: convective towers in front of stratiform rain (Houze 1989). Because the scaling in either region leads to similar conclusions, only the stratiform rain region will be considered. For this region consider the parameters

$$L = 100 \text{ km}, \qquad D = 10 \text{ km}, \tag{3.1a}$$
$$U = 10 \text{ m s}^{-1}, \qquad W = 1 \text{ m s}^{-1},$$
$$S_1 = 10^{-3}, \tag{3.1b}$$

where the advective timescale T is on the order of a few hours. Houze (1989) states that the maximum vertical velocity in this part of an MCS is approximately 0.5 m s^{-1}. While nonintegral values for the vertical velocity may be used, the value above leads to a simpler scaled system and the same conclusions. Substituting the above scaling parameters into (2.4), the dimensionless system that describes slowly varying mesoscale motions near the equator is

$$\frac{ds}{dt} - \varepsilon^{-2}\tilde{s}(w - H) = 0, \tag{3.2a}$$

$$\frac{du}{dt} + \rho_0^{-1}p_x - \varepsilon^2 yv = 0, \tag{3.2b}$$

$$\frac{dv}{dt} + \rho_0^{-1}p_y + \varepsilon^2 yu = 0, \tag{3.2c}$$

$$\frac{dw}{dt} + \varepsilon^{-2}(\rho_0^{-1}p_z + gs) = 0, \tag{3.2d}$$

$$\frac{dp}{dt} + \varepsilon^{-3}\gamma p_0(u_x + v_y + w_z) = 0, \tag{3.2e}$$

where $d/dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$ and $\varepsilon = 10^{-1}$. (In the BDT, once a standard nondimensionalization is performed and appropriate scaling parameters chosen for the flow of interest, dimensionless quantities such as the Froude or Rossby number are replaced by the appropriate power of ε for application of mathematical tools.) Here the heating term H must necessarily be added in order for a slowly evolving solution to exist (Browning and Kreiss 1997). Note that the only difference between the equatorial and midlatitude cases (Browning and Kreiss 1997) is the small size of the Coriolis term (the small size is due to the fact that at the equator the Coriolis term is zero).

Under the assumption that there is a slowly evolving component of a midlatitude mesoscale storm forced by cooling and heating that is essentially independent of gravity waves, Browning and Kreiss (1994, 1997) proved that the corresponding component of the solution is accurately described by a reduced system. The reduced system for the equatorial case (3.2) is

$$\frac{du}{dt} + \rho_0^{-1}p_x = 0, \tag{3.3a}$$

$$\frac{dv}{dt} + \rho_0^{-1}p_y = 0, \tag{3.3b}$$

$$u_x + v_y = RF_p, \tag{3.3c}$$

$$w = H, \tag{3.3d}$$

where $d/dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + H\partial/\partial z$, $F_p = -H_z$, and R is a projection operator (discussed below). This system is derived by making use of the fact that the vertical velocity w in (3.2e) can be replaced by H because of the balance that must exist between w and H for these flows [see (3.2a)]. In this system the vertical component of vorticity $\zeta = -u_y + v_x$ is generated only by the terms $(u_x + v_y)\zeta$, $-H_y u_z$, and $H_x v_z$ in the time-dependent equation for ζ. This conclusion should be contrasted with the midlatitude reduced system, where the Coriolis term plays a major role in the production of the vorticity.

In the case that the lateral dimensions of the domain are the same as the heating, the projection operator R just removes the mean of F_p. This is a natural restriction on F_p given that the left-hand side of (3.3c) is the divergence of the horizontal velocity. However, in the more realistic case when the lateral dimensions of the domain are much larger than the heating, R is a projection operator that removes the long horizontal waves of F_p (Browning and Kreiss 1997). To better understand why the projection operator R in the latter case has this form, consider the linear version of (3.2):

$$s_t - \varepsilon^{-2}\tilde{s}w = -\varepsilon^{-2}\tilde{s}H, \tag{3.4a}$$

$$u_t + \phi_x = 0, \tag{3.4b}$$

$$v_t + \phi_y = 0, \tag{3.4c}$$

$$w_t + \varepsilon^{-2}(\phi_z + gs) = 0, \tag{3.4d}$$

$$\phi_t + \varepsilon^{-3}C^2(u_x + v_y + w_z) = 0, \tag{3.4e}$$

where $\phi = p/\rho_0$ and for simplicity the mean state is assumed to be an isothermal atmosphere at rest. Because the mean state is isothermal, the stratification parameter $\tilde{s}$ and the speed of sound $C^2 = \gamma p_0/\rho_0$ are constants, that is, system (3.4) is a constant coefficient system of partial differential equations.

System (3.4) has a component of the solution that is slowly evolving in time and can be obtained by using the BDT (Kreiss 1979, 1980). Starting from rest, consider (3.4) on the domain $-\infty \le x, y < \infty$ and $0 \le z \le \pi$ with solid wall boundary conditions at the bottom and top of the atmosphere [$w(x, y, 0, t) = w(x, y, \pi, t) = 0$]. Here the domain has been chosen to have infinite extent in the horizontal directions so that the gravity waves will behave more as they do in reality (Browning and Kreiss 1997). For horizontal wavenumbers that are $O(1)$ in this domain, that is, for motions with the same length scale as the heating, the scaling above is valid. As in the original theory, differentiate (3.4e) with respect to t and replace u_{xt} and v_{yt} by differentiating (3.4b) with respect to x and (3.4c) with respect to y to obtain

$$\phi_{tt} - \varepsilon^{-3}C^2(\phi_{xx} + \phi_{yy}) = -\varepsilon^{-3}C^2 w_{zt}. \tag{3.5}$$

Now use the hydrostatic relation [assuming there is very little energy in the sound waves, the term w_t in (3.4d) can be neglected] to replace s in (3.4a), solve the resulting expression for w, and replace the term w_{zt} in Eq. (3.5) using the expression for w to derive the equation for ϕ:

$$\left[-C^2(g\tilde{s})^{-1}\frac{\partial^2}{\partial z^2} + \varepsilon\right]\phi_{tt} - \varepsilon^{-2}C^2(\phi_{xx} + \phi_{yy}) = -\varepsilon^{-2}C^2 H_{zt}. \tag{3.6}$$

Equation (3.6) in conjunction with (3.4b)–(3.4c) becomes the system

$$u_t + \phi_x = 0, \tag{3.7a}$$

$$v_t + \phi_y = 0, \tag{3.7b}$$

$$\phi_{zztt} + \varepsilon^{-2}g\tilde{s}(\phi_{xx} + \phi_{yy}) = \varepsilon^{-2}g\tilde{s}H_{zt}, \tag{3.7c}$$

where the small term $\varepsilon\phi_{tt}$ in (3.6) has been neglected.

[The reason that it is permissible to drop the small terms is that it can be proved that if the initial data for (3.4) are chosen so that a number of initial space and time derivatives are $O(1)$, then the ensuing space and time derivatives will be $O(1)$. Thus for solutions of this type, terms multiplied by a factor of ε will remain small.] The equation for ϕ can be solved independently of the other two, and then the solution ϕ becomes a forcing term in the first two equations.

First consider Eqs. (3.7c) for wavenumbers $O(1)$. The solution of the homogeneous equation varies on the fast timescale $O(\varepsilon^{-1})$, but there is a slowly evolving solution that can be obtained as accurately as desired by bounding higher- and higher-order time derivatives. For example, the first-order time derivative of (3.7c) for these waves is $O(1)$ if

$$\phi_{xx} + \phi_{yy} = H_{zt}. \tag{3.8}$$

Note that this part of the solution is accurately described by the linear version of the reduced system.

Now consider motions that have a much larger lateral scale than the heating, that is, those whose wavenumbers are $O(\varepsilon)$. To obtain the appropriate scaling for these waves the substitution $\tilde{x} = \varepsilon x$ in (3.7c) is appropriate. The correct scaling for these waves is

$$\phi_{zztt} + g\tilde{s}(\phi_{\tilde{x}\tilde{x}} + \phi_{\tilde{y}\tilde{y}}) = g\tilde{s}G, \tag{3.9}$$

where the fact that the energy in the heating for wavenumbers $O(\varepsilon)$ is proportional to the area of the wavenumber plane containing those wavenumbers, that is, $H_{zt} = \varepsilon^2 G$, has been used (Browning and Kreiss 1997). From (3.9), it can be seen that for the waves with a much longer length scale than the heating, the system does not have multiple timescales and the expansion of this part of the solution in terms of the reduced system is inappropriate. Thus these waves cannot be included in the reduced system and this explains the need for the projection operator R in (3.3). Note that this part of the solution consists of gravity waves with the same timescale (even after heating is shut off) and the same size pressure perturbations (until they spread out sufficiently far from the heating) as the dominant solution.

If one applies the analysis used in the midlatitude case to the equatorial mesoscale system (3.2) using (3.3) as the reduced system, similar conclusions concerning the solution will be obtained. In particular, the gravity waves generated by an equatorial mesoscale storm will have a timescale on the order of a few hours, a length scale $O(1000$ km), a depth scale $O(10$ km), pressure perturbations $O(10^2$ kg m^{-1} s$^{-2})$, horizontal velocity $O(1$ m s$^{-1})$, and vertical velocity $O(0.1$ m s$^{-1})$, that is, exactly as in the midlatitudes.

4. Slowly evolving large-scale motions near the equator

For slowly evolving large-scale equatorial flows with a timescale $O(1$ day), the relevant scaling parameters are

$$L = 1000 \text{ km}, \qquad D = 10 \text{ km}, \tag{4.1a}$$

$$U = 10 \text{ m s}^{-1}, \qquad W = 10^{-1} \text{ m s}^{-1},$$

$$S_1 = 10^{-3}, \tag{4.1b}$$

and substituting these into (2.4) the relevant dimensionless system is

$$\frac{ds}{dt} - \varepsilon^{-2}\tilde{s}(w - H) = 0, \tag{4.2a}$$

$$\frac{du}{dt} + \rho_0^{-1}p_x - yv = 0, \tag{4.2b}$$

$$\frac{dv}{dt} + \rho_0^{-1}p_y + yu = 0, \tag{4.2c}$$

$$\frac{dw}{dt} + \varepsilon^{-4}(\rho_0^{-1}p_z + \varepsilon\tilde{p}p + gs) = 0, \tag{4.2d}$$

$$\frac{dp}{dt} + \varepsilon^{-3}wp_{0z} + \varepsilon^{-3}\gamma p_0(u_x + v_y + w_z) = 0, \tag{4.2e}$$

where $d/dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$. To ensure the existence of a slowly evolving solution, a heating term has been added to (4.2a). [See Lin and Johnson (1996) for a discussion of heating rates $O(10°$ day$^{-1})$ that could lead to this size of heating.] For this large-scale case, very large-scale [$L = O(10\,000$ km)] gravity waves with a 1-day timescale and velocity components an order of magnitude smaller than in the dominant component can be expected to originate in these storms.

It is interesting to note that according to a number of studies (for a review of these studies, see Madden and Julian 1994), equatorial surface pressure waves with a 30–60-day period have been observed at stations near the equator. For slowly moving large-scale [$L = O(1000$ km)] equatorial storms with a heating timescale longer than a day [e.g., see the discussion of heating with such periods in Cadet (1986) or Knutson and Weickmann (1987)], the balanced solution that can be generated can have a period much longer than a day. To see how this is possible, it is necessary to change the timescale T of the flow. Clearly the timescale can be increased either by increasing L or decreasing U. Here the length scale is already appropriate for a monsoon, so the scale of the horizontal velocity will be reduced by an order of magnitude [e.g., see Knutson and Weickmann (1987) for a discussion of outgoing longwave radiation (OLR) anomalies in the Tropics with similar speeds]. Thus the relevant scaling parameters for large-scale equatorial flows with a timescale $O(10$ days) are

$$L = 1000 \text{ km}, \qquad D = 10 \text{ km}, \tag{4.3a}$$

$$U = 1 \text{ m s}^{-1}, \qquad W = 10^{-3} \text{ m s}^{-1},$$

$$S_1 = 10^{-4}. \tag{4.3b}$$

Note that the small velocities associated with this type of motion would be very difficult to observe. However, also note that the pressure perturbations and vertical velocity can be up to an order of magnitude larger than indicated in (4.3) if the flow is skewed, that is, if the velocity in the x direction is much larger than the velocity in the y direction. An example of elongated heating can be seen in plots of the OLR (Wheeler and Kiladis 1999, their Fig. 7). Also there are indications of vertical velocity of this larger size in numerical models (e.g. Hoskins et al. 1999, their Fig. 2).

Substituting the scaling parameters (4.3) into (2.4), the nondimensional system that describes large-scale equatorial flows with a timescale $O(10 \text{ days})$ is

$$\frac{ds}{dt} - \varepsilon^{-2}\tilde{s}(w - H) = 0, \tag{4.4a}$$

$$\frac{du}{dt} + \varepsilon^{-1}(\rho_0^{-1}p_x - yv) = 0, \tag{4.4b}$$

$$\frac{dv}{dt} + \varepsilon^{-1}(\rho_0^{-1}p_y + yu) = 0, \tag{4.4c}$$

$$\frac{dw}{dt} + \varepsilon^{-6}(\rho_0^{-1}p_z + gs) = 0, \tag{4.4d}$$

$$\frac{dp}{dt} + \varepsilon^{-4}\gamma p_0(u_x + v_y + \varepsilon w_z) = 0, \tag{4.4e}$$

where $d/dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + \varepsilon w\partial/\partial z$ and again a heating term [equivalent to a heating rate $O(0.1° \text{ day}^{-1})$] has been added to (4.4a) to ensure the existence of a slowly evolving solution. For slowly evolving in time solutions, the vertical velocity will be in balance with the heating and the pressure will satisfy the nonlinear balance equation with the horizontal divergence δ replaced by $-H_z$. There has been independent confirmation (J. Whitaker 1999, personal communication) that the use of large-scale heating with an $O(10 \text{ days})$ timescale behaves very differently than the same heating with an $O(1 \text{ day})$ timescale; that is, in the former case the flow is in geostrophic balance and in the latter it is not.

5. Normal mode expansion and initialization of equatorial solutions

A number of studies have expanded the solution of the diabatically forced system for the equator or equatorial observational data in the presence of heating into a series of normal modes of the linear adiabatic system (e.g., Silva Dias et al. 1983; Milliff and Madden 1996). Now it is possible to understand a number of problems with the normal mode series expansion approach that has been used previously using the linear system (3.4). In the same domain as discussed in section 3, Eq. (3.7c) can be transformed in the z direction to spectral space using a Fourier cosine series in z. Denoting the wavenumber corresponding to z by m, the transformed equation is

$$-m^2\hat{\phi}_{tt} + \varepsilon^{-2}g\tilde{s}(\hat{\phi}_{xx} + \hat{\phi}_{yy}) = \varepsilon^{-2}g\tilde{s}im\hat{H}_t, \tag{5.1}$$

where the hat notation indicates the transform coefficient for a variable and $\hat{H} = \hat{H}(x, y, m, t)$. Here it is assumed that the heating is zero at the bottom and top of the atmosphere. Then H is expandable in a Fourier sine series and the right-hand side of (3.7c) is representable as a Fourier cosine series. For example, for a typical heating function like $H = H'(x, y, t)\sin(z)$ (e.g., see Houze 1989), $H_{zt} = H_t'(x, y, t)\cos(z)$ and $m = 1$. If the scaling has been done correctly, then $\hat{H}_t(x, y, 1, t) = O(1)$ and Eq. (3.7c) maintains the same scaling in spectral space as in physical space. Note that the magnitude of the frequency of the homogeneous solution of (5.1) is $O(\varepsilon^{-1})$, that is, 10 times larger than that of the slow solution. A number of authors (e.g., Gill 1980; Silva Dias et al. 1983; Milliff and Madden 1996) have used only the first internal mode in their analyses. Of course if we used different values for m, different results would be obtained. In particular, if we assume $m = \varepsilon^{-1/2}$, then (5.1) becomes

$$\hat{\phi}_{tt} + \varepsilon^{-1}g\tilde{s}(\hat{\phi}_{xx} + \hat{\phi}_{yy}) = \varepsilon^{-3/2}g\tilde{s}i\hat{H}_t, \tag{5.2}$$

where small terms have been neglected. Equation (5.2) is essentially the gravity wave equation for the internal mode shallow-water equations at the equator driven by forcing. There are two possible problems here. The first is that as m increases, the heating becomes the dominant term in (5.1) with nothing else to balance it; that is, there is no balanced state for higher wavenumbers unless the heating coefficient for those waves is smaller than that for the lower wavenumbers. But even if the heating has a discontinuity in z, it will fall off as $1/m$, which is sufficiently fast to preclude this problem. The second is that the magnitude of the frequency of the fast waves $[O(\varepsilon^{-1/2})]$ is only about three times larger than the magnitude of the frequency of the slow solution. [Note that if the size of the horizontal velocities given in (2.1) were decreased, this ratio would increase.] As shown by Browning and Kreiss (1987), when the frequencies of the waves are all the same size (in this case, all slow), the system no longer has multiple timescales. And when the system is close to this limit case, higher-order derivatives must be bounded to provide an adequate initialization.

A major problem with the normal mode series expansion concept is that one tends to think in terms of the modes of the linearized adiabatic system. However, when one adds forcing (heating) to a system, it has been shown that the nature of the balanced diabatic solution is determined by the forcing and has only a very remote relationship to the behavior of the individual normal modes of the adiabatic system. To see this problem more clearly, write system (3.4) in the block form

$$\zeta_t = 0,$$

$$\zeta = -u_y + v_x, \tag{5.3a}$$

$$s_t - \varepsilon^{-2}\tilde{s}w = \varepsilon^{-2}\tilde{s}H, \tag{5.3b}$$

$$\delta_t + \nabla^2\phi = 0,$$

$$\delta = u_x + v_y, \quad (5.3c)$$

$$w_t + \varepsilon^{-2}(\phi_z + gs) = 0, \quad (5.3d)$$

$$\phi_t + \varepsilon^{-3}C^2(u_x + v_y + w_z) = 0, \quad (5.3e)$$

where it should be noted that this is equivalent to separating the slow and fast variables of the adiabatic problem as is done in nonlinear normal mode initialization (e.g., Machenhauer 1977). Here it is clear that the forcing only appears in the ''fast variable'' equations and that when a normal mode expansion of the solution is performed, the fast variable modes will have large coefficients even when the solution is slowly evolving in time. Stated in another manner, this means that just because the coefficients of modes associated with the fast variables of the adiabatic problem are large, it is not necessarily the case that there is large energy in the component of the diabatic solution evolving on the fast timescale.

It should also be pointed out that as a result of the thinking discussed above, the simple form of nonlinear normal mode initialization (Machenhauer 1977) does not work when heating is dominant. In that case setting the first-order time derivatives of the fast modes to zero yields the initial values

$$w = H \quad (5.4b)$$

$$\nabla^2\phi = 0, \quad (5.4c)$$

$$\phi_z + gs = 0, \quad (5.4d)$$

$$u_x + v_y + w_z = 0. \quad (5.4e)$$

At this point there is a $O(1)$ error in the initial values because for the dominant component $\delta_t = O(1)$ [see (3.8)].

6. Numerical examples

A number of the theoretical results in the previous sections can be demonstrated by specifying a heating function $H(x, y, z, t)$ that has spatial and temporal distributions similar to those observed in the atmosphere. For the equatorial mesoscale case, choose $H = H_0H_1(x, y)H_2(z)H_3(t)$, where

$$H_0 = 0.5 \text{ m s}^{-1}, \quad (6.1a)$$

$$H_1(x, y) = \exp\{-[(x - L/2)^2 + (y - L/2)^2]/r_e^2\},$$

$$r_e = 50 \times 10^3 \text{ m}, \quad (6.1b)$$

$$H_2(z) = \sin(\pi z/D), \quad (6.1c)$$

$$H_3(t) = \exp[-(t - 6 \times 3600)^2/(2 \times 3600)^2], \quad (6.1d)$$

$L = 1000 \times 10^3$ m is the length and width of the domain (the lateral dimensions of the domain are intentionally chosen to be considerably larger than that of the heating to allow the gravity waves to behave more as they would in the real atmosphere), and $D = 12 \times 10^3$ m is the depth. The spatial component of the heating $H_0H_1H_2$ (m s^{-1}) at $z = 6$ km is shown in Fig. 1. Houze states that the magnitude of the vertical velocity in a wide range of MCSs is on the order of 0.1–0.5 m s^{-1}. From (2.2a), the maximum vertical velocity is determined by the maximum value of H and from (6.1a) that will be 0.5 m s^{-1}. The time dependence of the heating (6.1d) is a Gaussian distribution centered at 6 h with an e-folding parameter of 2 h.

In earlier work (Browning and Kreiss 1997) the accuracy of the reduced system was shown by computing the norm of the difference of the solutions of the full and reduced systems divided by the norm of the solution of the full system and showing that this quantity is small as predicted by the theory. Here it will suffice to show that in the full system model (see above reference for description of this model) at the time of maximum heating the vertical velocity has the same shape and size of the heating (Fig. 2) and that shortly after the heating is turned off ($t = 18$ h) the vertical velocity of the gravity waves is small and is of much larger scale than the heating (Fig. 3).

For the large-scale equatorial case the heating is chosen as

$$H_0 = 0.05 \text{ m s}^{-1}, \quad (6.2a)$$

$$H_1(x,y) = \exp\{-[(x - L/2)^2 + (y - L/2 + 1.2 \times 10^6)^2]/r_e^2\},$$

$$r_e = 750 \times 10^3 \text{ m}, \quad (6.2b)$$

$$H_2(z) = \sin(\pi z/D), \quad (6.2c)$$

$$H_3(t) = \exp[-(t - 12 \times 3600)^2/(6 \times 3600)^2], \quad (6.2d)$$

where $L = 20\,000 \times 10^3$ m is the length and width of the domain, and $D = 12 \times 10^3$ m is the depth. The spatial component of the heating $H_0H_1H_2$ (m s^{-1}) at $z = 6$ km is shown in Fig. 4. Note that the spatial portion of the large-scale heating is chosen to match that for the Amazon basin used in Silva Dias et al. (1983) and that the maximum vertical velocity for this case will be 0.05 m s^{-1}. The time dependence of the heating (6.2d) is a Gaussian distribution centered at 12 h with an e-folding parameter of 6 h similar to the time dependence in Silva Dias et al. (1983). As in the first example, the vertical velocity at the time of maximum heating has the same shape and size of the heating (Fig. 5), and shortly after the heating is turned off ($t = 36$ h) the vertical velocity of the gravity waves is small and is of much larger scale than the heating (Fig. 6). Also note that a case with double the heating period produced gravity waves with a period on the order of a few days.

The results from the latter numerical example should be compared with earlier work on low-frequency equatorial waves (Gill 1980; Heckley and Gill 1984). The differences can be explained in the following manner.

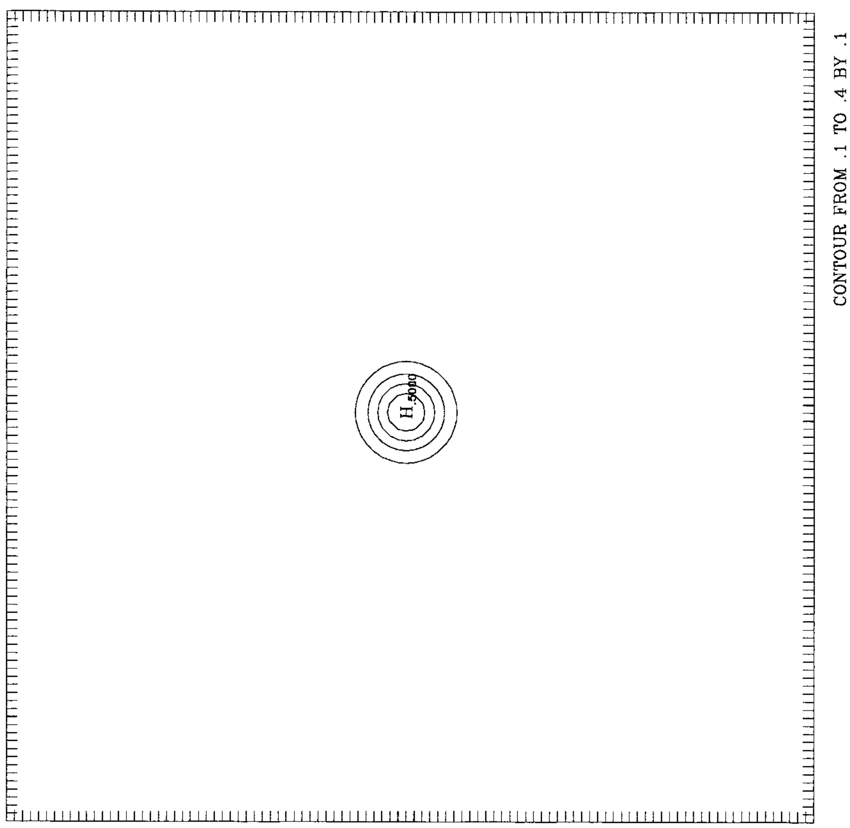

FIG. 1. The spatial component of the heating $H_0H_1H_2$ at $z = 6$ km for the equatorial mesoscale case. The length of a side of the square domain is 1000 km and the contour interval is 0.1 m s^{-1}.

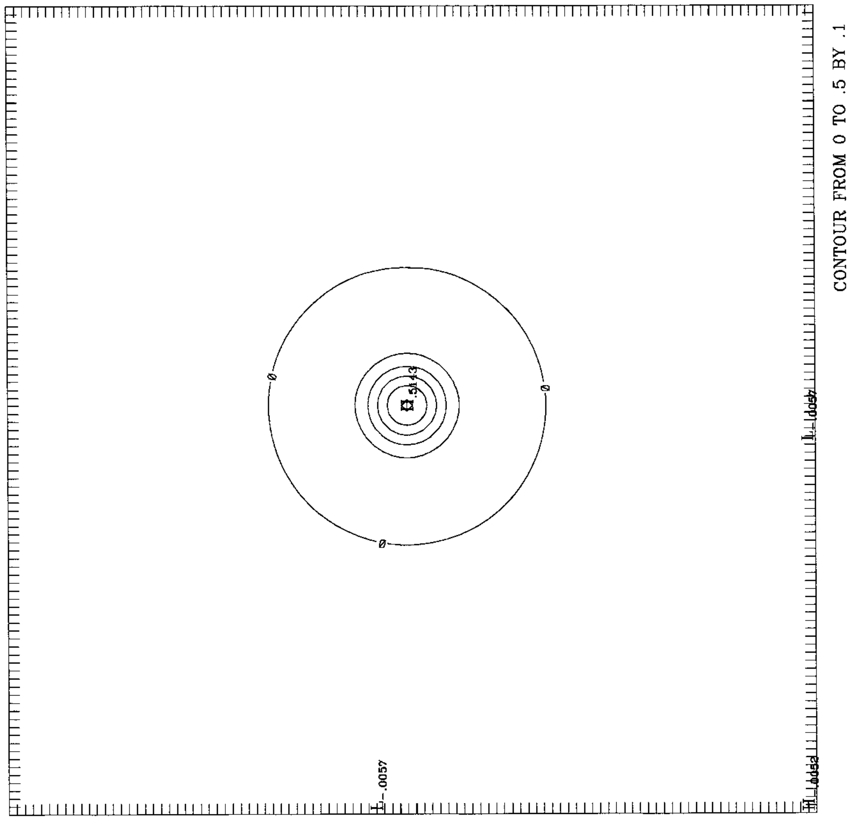

FIG. 2. The vertical velocity at $z = 6$ km and $t = 6$ h for the equatorial mesoscale case. The length of a side of the square domain is 1000 km and the contour interval is 0.1 m s^{-1}.

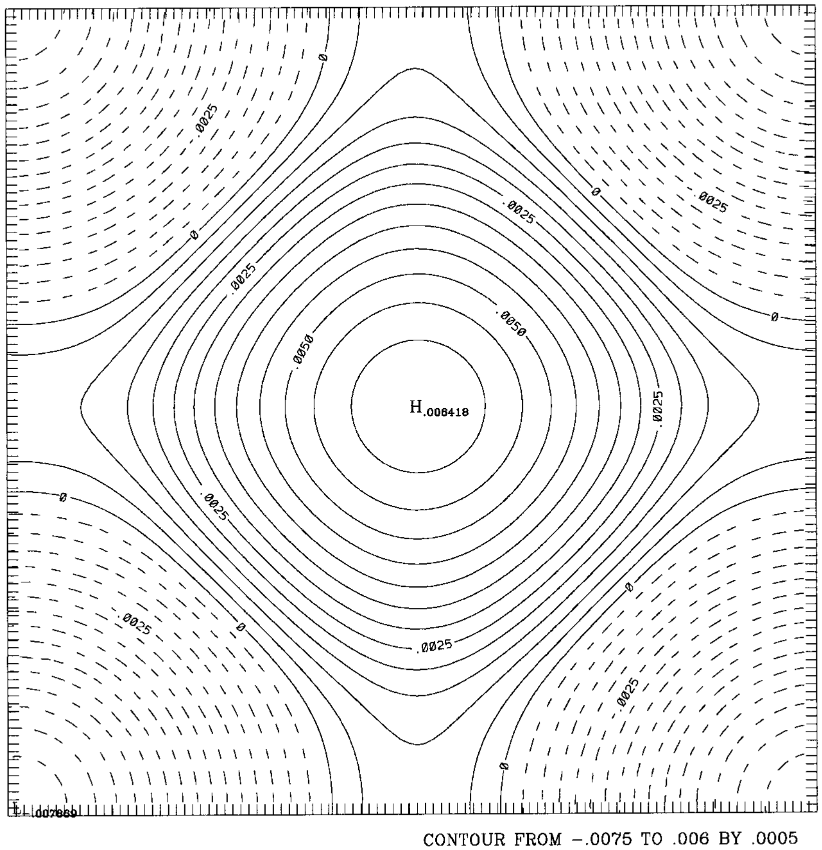

FIG. 3. The vertical velocity at $z = 6$ km and $t = 12$ h for the equatorial mesoscale case. The length of a side of the square domain is 1000 km and the contour interval is 0.0005 m s^{-1}.

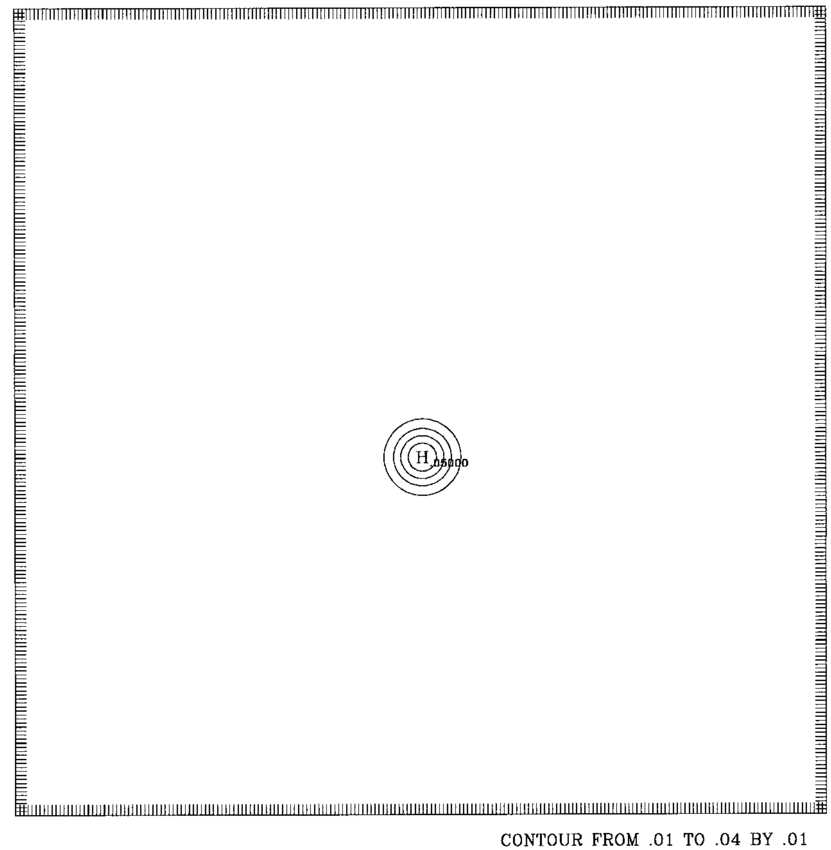

FIG. 4. The spatial component of the heating $H_0H_1H_2$ at $z = 6$ km for the equatorial large-scale case. The length of a side of the square domain is 20 000 km and the contour interval is 0.01 m s^{-1}.

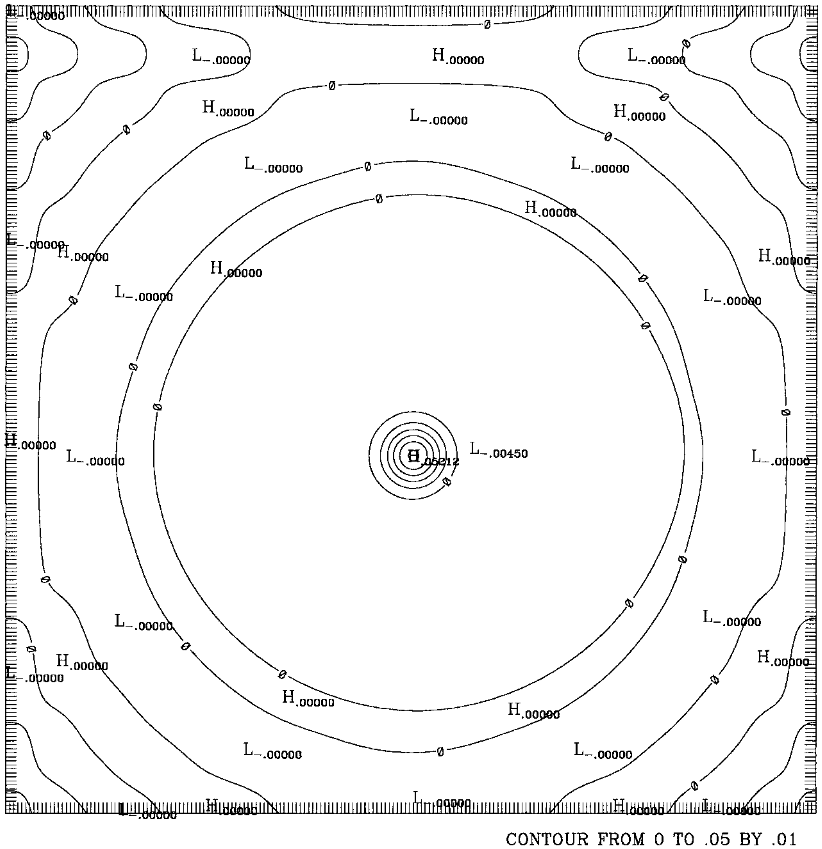

FIG. 5. The vertical velocity at $z = 6$ km and $t = 12$ h for the equatorial large-scale case. The length of a side of the square domain is 20 000 km and the contour interval is 0.01 m s^{-1}.

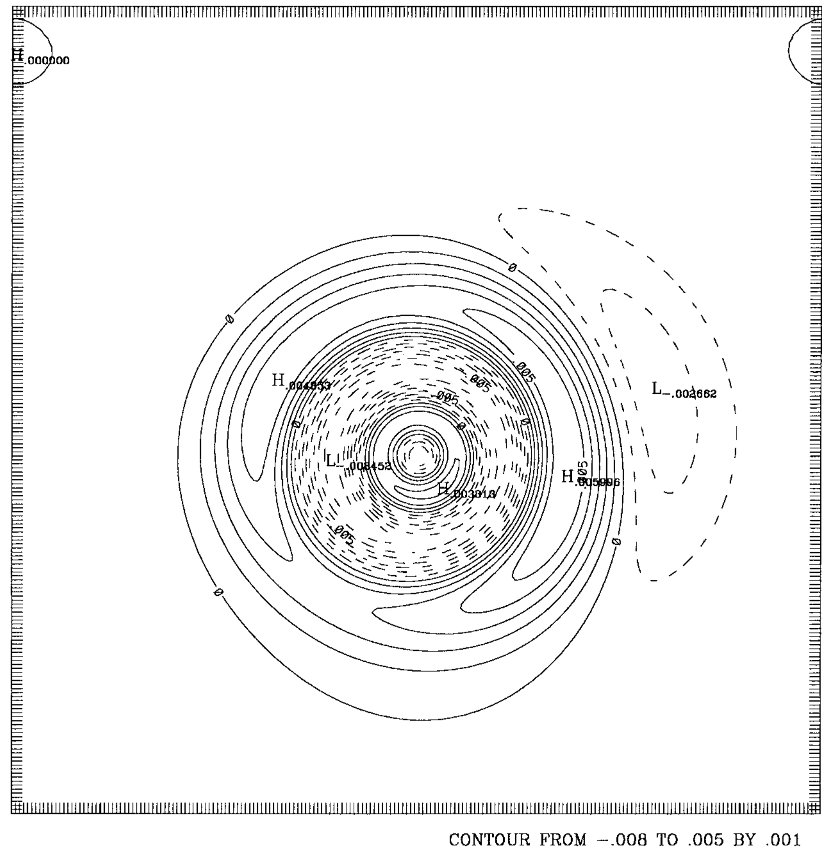

FIG. 6. The vertical velocity at $z = 6$ km and $t = 36$ h for the equatorial large-scale case. The length of a side of the square domain is 20 000 km and the contour interval is 0.001 m s^{-1}.

In the earlier work the scaling parameter for the horizontal direction was chosen to be the equatorial Rossby radius $\tilde{R} = (c/2\beta)^{1/2}$, where $c = (g\tilde{D})^{1/2}$ is the phase speed, $g = 9.8$ m s^{-2} is the gravitation, $\tilde{D} = 400$ m is the equivalent depth, and $\beta = 2 \times 10^{-11}$ is the Coriolis variation factor, so $\tilde{R} \approx 1250$ km. Heckley and Gill used an e-folding radius for the heating of two dimensionless units, that is, the heating had a radius O(3500 km), which is rather large (e.g., see Silva Dias et al. 1983) and near the limit of the accuracy of the beta-plane approximation. In addition, the heating was turned on instantaneously and left on (Heckley and Gill 1984). The time-invariant heat source would have caused the solution to become infinite without the addition of dissipation terms to the equations. Also, because the heating was left on, the Heckley and Gill solution contains both the dominant and gravity wave components of the solution. Even with the addition of viscosity, the large vertical velocity that is shown in the Heckley and Gill solution follows the heating as in the dominant component for the inviscid case. As a more realistic test, we used the heat source (6.2) with $r_e = 2000$ km. The equations (3.2) were Fourier cosine transformed in z (they then essentially become the linear shallow-water equations) and solved numerically using standard centered second-order finite difference approximations in space and time. The maximum magnitude of the geopotential perturbation was 0.8×10^3 m^2 s^{-2}, that is, as large as midlatitude geopotential perturbations. Given that geopotential perturbations of this size are not observed at the equator (also the horizontal divergence in this case did not follow the heating as closely as it did in the 3D large-scale case, which is to be expected for motions with larger geopotential perturbations because the balance between the horizontal divergence and the heating is less well satisfied), we also ran a test with $r_e = 750$ km as in the 3D large-scale case. Then the maximum magnitude of the geopotential perturbations was 2×10^2 m^2 s^{-2} and the divergence more closely followed the heating as expected. Slightly larger geopotential perturbations are observed at the equator, and the use of slightly larger heating rates as observed at the equator (Lin and Johnson 1996) with the more reasonable radius for heating produced more realistic geopotential perturbations.

7. Conclusions

It has been shown that slowly varying in time equatorial flows of any length scale satisfy the same balance between the vertical velocity and heating as in a midlatitude mesoscale storm forced by cooling and heating. Thus the gravity waves generated by equatorial storms will have the same time- and depth scales and the same size of pressure perturbations as the dominant component, a horizontal length scale an order of magnitude larger than that of the heating, and velocity components an order of magnitude smaller than the corresponding ones for the dominant component. In particular, for the large-scale case with a timescale O(1 day), the gravity waves will have a length scale O(10 000 km), and a timescale O(1 day). For slowly moving large-scale equatorial storms, for example, those with a heating time scale O(10 days), balanced pressure oscillations with a timescale O(10 days) can be produced.

Although a number of studies have shown large amounts of energy in gravity waves at the equator, this has been shown to be caused by the inappropriate projection of the dominant component of equatorial solutions onto the gravity modes of the linear adiabatic system. Because in reality there is little energy in the gravity wave portion of equatorial flows in the neighborhood of the storm, bounded derivative initialization can be applied without much effect on the solution. The details of the bounded derivative initialization for a limited area model that can be used anywhere on the globe and for any scale of motion will be discussed in a forthcoming paper.

Acknowledgments. The work of GLB was supported by the Office of Naval Research under Contract N00014-92-J-1089. The work of HOK was supported by the National Science Foundation under contract DMS 90-61311 and by the Office of Naval Research under Contract N00014-90-J-1382. The authors wish to express their appreciation to Ralph Milliff for his very constructive review and to Nita Fullerton for her technical editing.

REFERENCES

Baer, F., 1977: Adjustment of initial conditions required to suppress gravity oscillations in nonlinear flows. *Beitr. Phys. Atmos.,* **50,** 350–366.

——, and J. Tribbia, 1977: On complete filtering of gravity modes through nonlinear initialization. *Mon. Wea. Rev.,* **105,** 1536–1539.

Browning, G. L., and H.-O. Kreiss, 1986: Scaling and computation of smooth atmospheric motions. *Tellus,* **38A,** 295–313.

——, and ——, 1987: Reduced systems for the shallow water equations. *J. Atmos. Sci.,* **44,** 2813–2822.

——, and ——, 1994: The impact of rough forcing on systems with multiple timescales. *J. Atmos. Sci.,* **51,** 369–383.

——, and ——, 1997: The role of gravity waves in slowly varying in time mesoscale motions. *J. Atmos. Sci.,* **54,** 1166–1184.

——, A. Kasahara, and H.-O. Kreiss, 1980: Initialization of the primitive equations by the bounded derivative method. *J. Atmos. Sci.,* **37,** 1424–1436.

Cadet, D. L., 1986: Fluctuations of precipitable water over the Indian ocean during the 1979 summer monsoon. *Tellus,* **38A,** 170–177.

Charney, J. G., 1948: On the scale of atmospheric motions. *Geofys. Publ.,* **17,** 1–17.

——, 1955: The use of the primitive equations of motion in numerical prediction. *Tellus,* **7,** 22–26.

Garcia, R. R., and M. L. Salby, 1987: Transient response to localized episodic heating in the tropics. Part II: Far-field behavior. *J. Atmos. Sci.,* **44,** 499–530.

Gill, A. E., 1980: Some simple solutions for heat induced tropical circulation. *Quart. J. Roy. Meteor. Soc.,* **106,** 447–462.

Heckley, W. A., and A. E. Gill, 1984: Some simple analytical so-

lutions to the problem of forced equatorial long waves. *Quart. J. Roy. Meteor. Soc.,* **110,** 203–217.
Hinkelmann, K., 1951: Der mechanismus des meteorologischen Lärmes. *Tellus,* **3,** 285–296.
Hoskins, B., R. Neale, M. Rodwell, and G.-Y. Yang, 1999: Aspects of the large-scale tropical atmospheric circulation. *Tellus,* **51A-B,** 33–44.
Houze, R. A., 1989: Observed structure of mesoscale convective systems and implications for large-scale heating. *Quart. J. Roy. Meteor. Soc.,* **115,** 425–461.
Knutson, T. R., and K. M. Weickmann, 1987: 30–60 day atmospheric oscillations: Composite life cycles of convection and circulation anomalies. *Mon. Wea. Rev.,* **115,** 1407–1436.
Kreiss, H.-O., 1979: Problems with different time scales for ordinary differential equations. *SIAM J. Numer. Anal.,* **16,** 980–998.
——, 1980: Problems with different time scales for partial differential equations. *Commun. Pure Appl. Math,* **33,** 399–440.
Lin, X., and R. H. Johnson, 1996: Heating, moistening, and rainfall over the western Pacific warm pool during TOGA COARE. *J. Atmos. Sci.,* **53,** 3367–3383.
Machenhauer, B., 1977: On the dynamics of gravity oscillations in a shallow water model, with applications to normal mode initialization. *Beitr. Phys. Atmos.,* **50,** 253–271.
Madden, R. A., and P. R. Julian, 1994: Observations of the 40–50-day tropical oscillation—A review. *Mon. Wea. Rev.,* **122,** 814–837.
Milliff, R. F., and R. A. Madden, 1996: The existence and vertical structure of fast, eastward-moving disturbances in the equatorial troposphere. *J. Atmos. Sci.,* **53,** 586–597.
Phillips, N. A., 1956: The general circulation of the atmosphere: A numerical experiment. *Quart. J. Roy. Meteor. Soc.,* **82,** 123–164.
Salby, M. L., and R. R. Garcia, 1987: Transient response to localized episodic heating in the tropics. Part I: Excitation and short-time near-field behavior. *J. Atmos. Sci.,* **44,** 458–498.
Silva Dias, P. L., W. H. Schubert, and M. Demaria, 1983: Large-scale response of the tropical atmosphere to transient convection. *J. Atmos. Sci.,* **40,** 2689–2707.
Smagorinsky, J., 1963: General circulation experiments with the primitive equations. I. The basic experiment. *Mon. Wea. Rev.,* **91,** 99–164.
Wergen, W., 1988: The diabatic ECMWF normal mode initialisation scheme. *Beitr. Phys. Atmos.,* **61,** 274–302.
Wheeler, M., and G. N. Kiladis, 1999: Convectively coupled equatorial waves: Analysis of clouds and temperature in the wavenumber-frequency domain. *J. Atmos. Sci.,* **56,** 374–399.